NFSI IS NOT INCLUDED IN NF_3

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Abstract. We prove that the system of strong impredicative NF is not a part of a consistent 3-stratified extension of NF_3.

An instance of stratified comprehension $\exists y \forall x (x \in y \leftrightarrow A)$ is called strictly impredicative iff it can be stratified by assigning to x a type less than or equal to the type assigned to each other variable. NFSI is the fragment of Quine's NF based on extensionality and strictly impredicative stratified comprehension. This theory was introduced and proved consistent by S. Tupailo in [6].

Almost all extensional fragments of NF known to be consistent are subtheories of NFI (mildly impredicative NF), shown to be consistent in [3], or of NF_3 (the NF axioms with just three relative types), shown to be consistent in [5], or 3-stratified extensions of NF_3. Although NFSI seems to be quite weak, it is clearly not a subset of NFI, as NFSI satisfies the highly impredicative axiom of set union, which when added to NFI generates full NF. We show here that NFSI is also not a subtheory of a consistent 3-stratified extension of NF_3.

Definitions 1. A **Grishin structure** is a countable model $\mathfrak{M} = \langle M, \in_{\mathfrak{M}} \rangle$ of NF_2 (the 2-stratified axioms of NF) such that every infinite set (coded) in it can be cut into two disjoint infinite pieces.

A **Boolean automorphism** of a Grishin structure is an ordered pair of permutations $\langle \alpha_0, \alpha_1 \rangle$ of M such that $a \in_{\mathfrak{M}} b$ iff $\alpha_0(a) \in_{\mathfrak{M}} \alpha_1(b)$.

The previous definitions can be motivated as follows. An extensional structure $\mathfrak{M} = \langle M \in_{\mathfrak{M}} \rangle$ satisfying axioms asserting the existence of the singleton of a set, the union of two sets, and the complement of a set can be viewed as an atomic Boolean algebra (with singletons playing the role of atoms) and is in fact a model of NF_2 (see [1]). It has been shown in [5] that the Boolean algebras of Grishin structures are all isomorphic. Moreover, if $\langle \alpha_0, \alpha_1 \rangle$ is a Boolean automorphism of $\mathfrak{M}$, then α_1 is an automorphism of his Boolean algebra, and α_0 is the permutation induced by its restriction on the atoms: $\{\alpha_0(x)\} = \alpha_1(\{x\})$. Although we will use these definitions only for models of NF_3, they are also suitable for models of other extensions of NF_2, like NFSI, NFI, and NF.

We can easily express the existence of a bijection between disjoint sets by a formula stratified with 0, 1 and 2 by using unordered pairs instead of the usual

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ordered pairs, which would need one more type. Let $\mathsf{IC}(x^{[1]})$ be a formula[1], saying that the size of x is bigger than or equal to that of its complement by expressing that there is a set of unordered pairs establishing a bijection between a subset of x and the complement of x.

PROPOSITION 1. *For every model of* NF_3*, there is a 3-elementarily equivalent model* $\mathfrak{M}$*, such that* $\mathfrak{M} \models \mathsf{IC}(\mathsf{USC}(V))$*, i.e.* $\mathfrak{M}$ *satisfies the same 3-stratified sentences, and, in* $\mathfrak{M}$*, the set of all singletons* $\mathsf{USC}(V)$ *has at least as many elements as its complement!*

PROOF

We will show this by employing a technique of [4] used in type theory that we translate here in NF_3; see also [2], for a slightly different approach.

Let us take a countable model of the theory of the model with a nonstandard finite set. We then have an elementarily equivalent Grishin structure $\mathfrak{M} = \langle M \in_{\mathfrak{M}} \rangle$ in which there is an (externally) infinite noncofinite set satisfying $\mathsf{IC}(x)$. That this is the case if $\mathfrak{M}$ does not satisfy the axiom of infinity can be shown by considering a nonstandard set whose complement is nonstandard, and by noticing that all sets are comparable; and if $\mathfrak{M}$ satisfies the axiom of infinity it is equally obvious, because the complement of any nonstandard finite set will do.

If $a, b \in M$ are both infinite noncofinite sets, then using a familiar back and forth construction (see e.g. [5]) one obtains a Boolean automorphism $\alpha = \langle \alpha_0, \alpha_1 \rangle$ of $\mathfrak{M}$ with α_1 sending a to b. Define the model $\mathfrak{N}$ by solely modifying the $\in$-relation as follows:

$$y \in_{\mathfrak{N}} z \text{ by } y \in_{\mathfrak{M}} \alpha_0(z).$$

If $A(x^{[0]}, ..., y^{[1]}..., z^{[2]})$ is stratified with $0, 1, 2$, as indicated, we have, by induction on the length of A:

$$\mathfrak{N} \models A(a_0, ..., a_1, ..., a_2) \text{ iff } \mathfrak{M} \models A(a_0, ..., \alpha_0(a_1), ..., \alpha_1(\alpha_0(a_2)))$$

In particular, $\mathfrak{N}$ and $\mathfrak{M}$ verify the same 3-stratified sentences.

Notice that $\mathsf{USC}(V)^{\mathfrak{M}}$ is not finite, nor cofinite. Pick a in M such that $\mathfrak{M} \models \mathsf{IC}(a)$ and let α be a Boolean automorphism with α_1 sending a to $\mathsf{USC}(V)^{\mathfrak{M}}$. We thus have:

$$\mathfrak{N} \models \mathsf{IC}(\alpha_0^{-1}(a)) \text{ iff } \mathfrak{M} \models \mathsf{IC}(a)$$
$$\mathfrak{N} \models \alpha_0^{-1}(a) = \mathsf{USC}(V) \text{ iff } \mathfrak{M} \models \alpha_1(a) = \mathsf{USC}(V)$$

since the formulas $\mathsf{IC}(y^{[1]})$ and $z^{[2]} = \mathsf{USC}(V)$ are stratified with 0, 1 and 2.

Therefore $\mathfrak{N} \models \mathsf{IC}(\mathsf{USC}(V))$, as required. ∎

THEOREM 2. *If* S *is a 3-stratified consistent extension of* NF_3*, then* $S \not\vdash \mathsf{NFSI}_4$.

PROOF

Since proposition 1 entails that $S + \mathsf{IC}(\mathsf{USC}(V)))$ is consistent, the result will

[1]Bracketed superscripts are informal symbols that may be used to indicate the type index assigned by a specific stratification.

follow from the fact that $\mathsf{NFSI}_4 \vdash (\mathsf{USC}(V) \text{ exists} \rightarrow \neg\mathsf{IC}(\mathsf{USC}(V))$, which we now show.

In $\mathsf{NFSI}_4 + \mathsf{USC}(V)$ exists, the 4-stratified formula "$f^{[3]}$ is a bijection (coded by a set of unordered pairs) from a subset $\mathsf{USC}(u)$ of $\mathsf{USC}(V)$ onto its complement", implies the existence of the set $C = \{\, x^{[0]} \in u \;:\; x \notin f^{[3]}(\{x\}) \,\}$.

We show, in NFSI_4, that $C \notin \mathsf{USC}(V)$. We first observe that NFSI_4 proves the pairing axiom. We then suppose *per absurdum* that $C = \{a\}$, for some a, and we choose two elements b and c in u, distinct from a—this can be done because a model of NFSI_4 is infinite. If $f(\{x\}) = \{a, b\}$ then x cannot be a, because $a \notin f(\{a\})$. Hence, $x \in f(\{x\})$ and so x must be b, that is $f(\{b\}) = \{a, b\}$. Similarly $f(\{c\}) = \{a, c\}$. Again, it follows that $\{a, b, c\}$ must be $f(\{b\})$ or $f(\{c\})$, which is clearly impossible.

The usual proof of Cantor's theorem now goes through: let $d \in u$ such that $f(\{d\}) = C$; then $f(\{d\}) \in C$ iff $f(\{d\}) \notin C$. ∎

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