Cantor-Bernstein's theorem in a semiring

Marcel Crabbé

Consider a commutative semiring with sum $+$, product $\cdot$, the identity 1 for $\cdot$, and the identity 0 for $+$, which is absorbing for $\cdot$.

Suppose moreover that the semiring contains the element $\aleph_0$, and, to the usual axioms for commutative semirings, we add:

$$\aleph_0 = \aleph_0 + 1 \tag{1}$$

$$\text{if } \kappa + \mu \leq \kappa, \text{ then } \mu \cdot \aleph_0 \leq \kappa \tag{2}$$

Here the relation $\leq$ is defined by: $\kappa \leq \mu$ if and only if $\kappa + \nu = \mu$, for some ν. This relation is clearly transitive and reflexive.

Without being committed to implementation of cardinal numbers in a specific set theory, it is easily seen that all these axioms are satisfied in the usual interpretation, even without assuming the axiom of choice: the sum and the product of cardinal numbers are explained through union of disjoint sets and cartesian product, respectively; 0 is the cardinal of the empty set, 1 is the cardinal of a singleton, and $\aleph_0$ the cardinal of the set of natural numbers. I observe that according to this interpretation, $\kappa \leq \mu$ means that there is an injective function of a set of cardinal κ into a set of cardinal μ.

Finally, to see that axiom (2) is verified, let K, M be two disjoint sets of cardinal κ and μ, respectively. Let also f be an injective mapping from $K \cup M$ into K. Then to each x in M and natural number n, we associate the result of applying the $(n+1)$-fold iterate of f to x:

$$\begin{array}{c|ccccc}
 & & & K & & \\
M & f[M] & f[f[M]] & f[f[f[M]]] & & \\
\hline
x_1 & \to f(x_1) & \to f(f(x_1)) & \to f(f(f(x_1))) & \to & \ldots \\
x_2 & \to f(x_2) & \to f(f(x_2)) & \to f(f(f(x_2))) & \to & \ldots \\
x_3 & \to f(x_3) & \to f(f(x_3)) & \to f(f(f(x_3))) & \to & \ldots \\
 & & \vdots & & &
\end{array}$$

That is, the function g, defined by the following recursive equations

$$g(x, 0) = f(x)$$

$$g(x, n+1) = f(g(x, n))$$

is an injective mapping of $M \times \mathbb{N}$ into K.

Proposition 1 *If $\kappa + \mu \leq \kappa$, then $\kappa + \mu = \kappa$.*

Proof

If $\kappa + \mu \leq \kappa$, then $\mu \cdot \aleph_0 \leq \kappa$, by axiom (2). This means that, for some λ, $\mu \cdot \aleph_0 + \lambda = \kappa$. We then have

$$\begin{aligned} \kappa + \mu &= \mu \cdot \aleph_0 + \mu + \lambda \\ &= \mu \cdot (\aleph_0 + 1) + \lambda \\ &= \mu \cdot \aleph_0 + \lambda && \text{axiom (1)} \\ &= \kappa \end{aligned}$$

□

Cantor-Bernstein's theorem states that $\leq$ is a partial order:

if $\kappa \leq \mu$ and $\mu \leq \kappa$, then $\kappa = \mu$.

Proof

If $\mu \leq \kappa$, there exists λ such that $\mu + \lambda = \kappa$. Then, if $\kappa \leq \mu$, we have $\mu + \lambda \leq \mu$ and, by Proposition 1, $\mu + \lambda = \mu$, i.e., $\kappa = \mu$. □

Remarks

1. Axioms (1) and (2) are equivalent to the statement:

$$\kappa + \mu = \kappa \quad \text{if and only if} \quad \mu \cdot \aleph_0 \leq \kappa \tag{3}$$

which generalizes the well-known feature of Dedekind-infinite cardinals, namely,

$$\kappa + 1 = \kappa \qquad \text{if and only if} \qquad \aleph_0 \leq \kappa$$

2. $\aleph_0$ is the unique element satisfying (3).
3. The axioms allow one to derive directly the properties $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 \cdot \aleph_0 = \aleph_0$.

Indeed, as $\aleph_0 + 1 + 1 = \aleph_0$, we have $(1+1) \cdot \aleph_0 \leq \aleph_0$ and $\aleph_0 + \aleph_0 \leq \aleph_0$. Hence, by Proposition 1, $\aleph_0 + \aleph_0 = \aleph_0$. It follows that $\aleph_0 \cdot \aleph_0 \leq \aleph_0$. But $\aleph_0 \cdot \aleph_0 = (\aleph_0 + 1) \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 + \aleph_0$. Therefore, $\aleph_0 \leq \aleph_0 \cdot \aleph_0$ and, by Cantor-Bernstein's theorem, $\aleph_0 \cdot \aleph_0 = \aleph_0$.
4. Remark 1 and the fact that $\aleph_0 \cdot \aleph_0 = \aleph_0$ immediately entail

$$\kappa + \mu = \kappa \qquad \text{if and only if} \qquad \kappa + \mu \cdot \aleph_0 = \kappa$$

From this we get

$$\kappa + \kappa = \kappa \qquad \text{if and only if} \qquad \aleph_0 \cdot \kappa = \kappa$$