Math. Log. Quart. 40 (1994) 481 – 489

Mathematical Logic
Quarterly

The Hauptsatz for Stratified Comprehension: A Semantic Proof

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Abstract. We prove the cut-elimination theorem, Gentzen's Hauptsatz, for the system for stratified comprehension, i. e. Quine's NF minus extensionality.

Mathematics Subject Classification: 03B15, 03F05.

Keywords: Proof theory, New Foundations, Type theory, Cut elimination.

1 Introduction

In [2], we have proved that every natural-deduction derivation in the system for stratified comprehension, SF, reduces to a normal form. The proof was given for the system with underlying intuitionistic logic; it can be extended to classical logic and it can also be shown, using the equivalence between natural deduction and sequent calculus, that the corresponding sequent calculus admits cut-elimination.

We will give here a much simpler proof of cut-elimination for the classical sequent calculus for stratified comprehension. The idea of this kind of proof, which goes back to SCHÜTTE, is to prove the completeness of the cut-free system. Since the system with the cut-rule is complete too, this will show that both systems are the same. However this result is weaker than the one in [2], since it merely shows the existence of a cut-free derivation and provides no further information on the relation between the original derivation and the cut-free derivation.

The semantic proofs for type theory are found in [10] and [9]. One is also advised to read [11] and [6], for detailed presentations. In these papers one shows first the completeness of the the cut-free system with respect to partial valuations (models with truth value gaps) and then the non trivial part of the proof lies in showing that such a valuation always extends to a classical model. Our strategy will be somewhat different. We consider valuations without truth value gaps instead, and the non trivial part amounts to show that they have no truth value gluts (see [4], for a general setting). Our method can also be used to give an alternative proof of the Hauptsatz for type theory.

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2 Proof theory

We will describe here the system SF (Stratified Foundations) which is a version with terms of QUINE's NF (New Foundations) without extensionality.

The language of SF. The symbols used are $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$, $\in$, $\{\ \mid\ \}$, ( ) and a countably infinite set of variables. *Pseudo-terms*, *formulas* and *terms* are expressions, build up from these symbols, defined inductively as follows:

· every variable is a pseudo-term;

· atomic pseudo-formulas are $P \in Q$, where P and Q are pseudo-terms;

· if A, B are pseudo-formulas and x is a variable, then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$, $\forall x\, A$ and $\exists x\, A$ are pseudo-formulas, and $\{x \mid A\}$ is a pseudo-term.

A *weak stratification* for an expression (a pseudo-formula or a pseudo-term) is a function from the occurrences of pseudo-terms in that expression to the integers – called *types* – satisfying:

· at an occurrence of $P \in Q$, the type of P is i iff the type of Q is $i + 1$;

· in an occurrence of $\{x \mid A\}$, the type of each occurrence of x is the same and the type of $\{x \mid A\}$ is one higher;

· in an occurrence of $\forall x\, A$ or $\exists x\, A$, the type of each occurrence of x is the same.

A pseudo-formula or pseudo-term is *weakly stratifiable* iff there is a weak stratification for it. A pseudo-formula or pseudo-term is *stratifiable* iff there is a weak stratification for it such that all occurrences of a same variable have identical type.

A *term* is a weakly stratifiable pseudo-term (hence, variables are terms); *atomic formulas* are all of the form $P \in Q$, where P and Q are terms; if A, B are formulas and x is a variable, then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$, $\forall x\, A$ and $\exists x\, A$ are *formulas*.

Note that, although a term is always weakly stratifiable, a formula need not be weakly stratifiable: the pseudo-term $\{x \mid x \in x\}$ is *not* a a term, though $x \in x$ is weakly stratifiable; but $\exists x\, x \in x$ is a formula.

Bound and free variables, substitution. The variable binding operators are $\forall$, $\exists$ and $\{\ \mid\ \}$. If A is a formula, x a variable, P a term and no free variable of P is bound in A, then $A[x := P]$ is the result of substituting P for x in A; $Q[x := P]$ is defined similarly for terms Q and P.

We will speak as usual of free and bound occurrences of variables, but we will actually identify terms or formulas that differ only up to renaming of bound variables. Thus, strictly speaking, we redefine the notions of terms and formulas as being equivalence classes of what we have heretofore called terms and formulas. The result is that bound variables no longer really occur: this procedure amounts roughly to Bourbaki's method or to introducing De Bruijn indices.

The sequent calculus. A *sequent* is an ordered pair of finite sets of formulas. The sequent $\langle \Gamma, \Delta \rangle$ is denoted by $\Gamma \vdash \Delta$. When dealing with sequents we may write Γ, Δ for $\Gamma \cup \Delta$ and A for $\{A\}$.

Initial sequents: $\Gamma \vdash \Delta$ is an *initial sequent* iff $\Gamma \cap \Delta \neq \emptyset$.

Introduction rules:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg_R$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A, \Delta; \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta; \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash A, \Delta; \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\ \rightarrow_L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\ \rightarrow_R$$

$$\frac{\Gamma \vdash A, B, \Delta; \quad \Gamma, A, B \vdash \Delta}{\Gamma, A \leftrightarrow B \vdash \Delta}\ \leftrightarrow_L \qquad \frac{\Gamma, A \vdash B, \Delta; \quad \Gamma, B \vdash A, \Delta}{\Gamma \vdash A \leftrightarrow B, \Delta}\ \leftrightarrow_R$$

$$\frac{\Gamma, A[x := P] \vdash \Delta}{\Gamma, \forall x\, A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[x := y], \Delta}{\Gamma \vdash \forall x\, A, \Delta}\ \forall_R$$

$$\frac{\Gamma, A[x := y] \vdash \Delta}{\Gamma, \exists x\, A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[x := P], \Delta}{\Gamma \vdash \exists x\, A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A[x := P] \vdash \Delta}{\Gamma, P \in \{x \mid A\} \vdash \Delta}\ \in_L \qquad \frac{\Gamma \vdash A[x := P], \Delta}{\Gamma \vdash P \in \{x \mid A\}, \Delta}\ \in_R$$

Restrictions: In instances of the rules $\forall_R$ (or $\exists_L$), the *proper variable* y does not occur (i.e., the variable y does not occur free in the usual sense) in the formulas in Γ, Δ, $\forall x\, A$ (or Γ, Δ, $\exists x\, A$).

Cut rule:

$$\frac{\Gamma \vdash A, \Delta; \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}.$$

A sequent is *derivable* [*cut-free derivable*] iff there is a derivation starting from initial sequents, using the rules [introduction rules] and ending with the sequent.

R e m a r k s .

(i) If one restricted oneself to stratifiable terms instead of allowing weakly stratifiable terms as well, the set of terms would not be closed under substitution; and the Hauptsatz would fail in a trivial way: for example, the derivable sequent

$$\vdash \forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge x \in z))$$

would not be cut-free derivable (see [5, p. 76).

(ii) For the same kind of reason, if one does not somehow identify formulas up to bound variables (as we did) or introduce two kinds of variables (free and bound) as GENTZEN did, a sequent like

$$\vdash y \in z \rightarrow \exists x \exists y (x \in z \wedge y \in z)$$

though derivable would not be cut-free derivable, but

$$\vdash \forall y (y \in z \rightarrow \exists x \exists y (x \in z \wedge y \in z))$$

would still be cut-free derivable.

W e a k e n i n g: One can prove that if a sequent $\Gamma \vdash \Delta$ is derivable [cut-free derivable], then all sequents $\Gamma, \Gamma' \vdash \Delta, \Delta'$ are derivable [cut-free derivable] too.

Let $F_0, F_1, \ldots$ be an effective enumeration of all the formulas of the language.

Lemma 1. *If the sequent $\Gamma \vdash \Delta$ is not cut-free derivable, there exists a sequence $\mathcal{G}$ of sequents $\Gamma_0 \vdash \Delta_0, \Gamma_1 \vdash \Delta_1, \ldots$ such that:*

(a) *no sequent in $\mathcal{G}$ is cut-free derivable;*

(b) *$\Gamma_0 \vdash \Delta_0$ is $\Gamma \vdash \Delta$;*

(c) *$\Gamma_i \subseteq \Gamma_{i+1}$ and $\Delta_i \subseteq \Delta_{i+1}$;*

(d) *if F_i is not of the form $\exists x\, A$ and $\Gamma_i \vdash F_i, \Delta_i$ is not cut-free derivable, then F_i belongs to Δ_{i+1};*

(e) *if F_i is $\exists x\, A$ and $\Gamma_i, F_i \vdash \Delta_i$ is not cut-free derivable, then F_i belongs to Γ_{i+1} and there is a term P such that $A[x := P]$ belongs to Γ_{i+1};*

(f) *if F_i is $\forall x\, A$ and $\Gamma_i \vdash F_i, \Delta_i$ is not cut-free derivable, then there is a term P such that $A[x := P]$ belongs to Δ_{i+1}.*

Proof. It will suffice to describe the transition from $\Gamma_i \vdash \Delta_i$ to $\Gamma_{i+1} \vdash \Delta_{i+1}$. Suppose that F_i is not of the form $\exists x\, A$ or $\forall x\, A$. If $\Gamma \vdash F_i, \Delta$ is not cut-free derivable, we let Γ_{i+1} be Γ_i and Δ_{i+1} be $\Delta_i \cup \{F_i\}$; else $\Gamma_{i+1} \vdash \Delta_{i+1}$ is $\Gamma_i \vdash \Delta_i$. Suppose F_i is $\exists x\, A$. If $\Gamma_i, \exists x\, A \vdash \Delta_i$ is not cut-free derivable, let y be the first variable not occurring free in one of the formulas of this sequent. We let Δ_{i+1} be Δ_i and Γ_{i+1} be $\Gamma_i \cup \{\exists x\, A, A[x := y]\}$; else $\Gamma_{i+1} \vdash \Delta_{i+1}$ is $\Gamma_i \vdash \Delta_i$. Finally, suppose that F_i is $\forall x\, A$. If $\Gamma_i \vdash \forall x\, A, \Delta_i$ is not cut-free derivable, and if y is the first variable not occurring free in one of the formulas of this sequent, we let Γ_{i+1} be Γ_i and Δ_{i+1} be $\Delta_i \cup \{\forall x\, A, A[x := y]\}$; else $\Gamma_{i+1} \vdash \Delta_{i+1}$ is $\Gamma_i \vdash \Delta_i$. □

Given $\Gamma \vdash \Delta$ and a sequence $\mathcal{G}$ verifying the clauses in Lemma 1, we define $\mathcal{G} \models^+ A$ as meaning that, for some i, $\Gamma_i \vdash A, \Delta_i$ is cut-free derivable; and $\mathcal{G} \models^- A$ as meaning that, for some i, $\Gamma_i, A \vdash \Delta_i$ is cut-free derivable.

The next lemma will show that a possible reading for $\mathcal{G} \models^+ A$ and $\mathcal{G} \models^- A$ is "A is true" and "A is false", respectively. Since we have not yet proved the Hauptsatz, it is not excluded at this stage that A be both true and false (the contradiction principle might be violated), however A is always true or false (excluded middle).

Lemma 2. *Let $\mathcal{G}$ be a sequence associated with $\Gamma \vdash \Delta$, satisfying the conditions of Lemma 1, then:*

(a) *$\mathcal{G} \not\models^- C$, for all C in Γ, and $\mathcal{G} \not\models^+ D$, for all D in Δ;*

(b) *$\mathcal{G} \models^+ A$ or $\mathcal{G} \models^- A$, for every formula A;*

and for every formulas B, C and variable x:

(c) *if $\mathcal{G} \models^+ B$, then $\mathcal{G} \models^- \neg B$;*

(d) *if $\mathcal{G} \models^- B$, then $\mathcal{G} \models^+ \neg B$;*

(e) *if $\mathcal{G} \models^+ B$ and $\mathcal{G} \models^+ C$, then $\mathcal{G} \models^+ B \wedge C$;*

(f) *if $\mathcal{G} \models^- B$ or $\mathcal{G} \models^- C$, then $\mathcal{G} \models^- B \wedge C$;*

(g) *if $\mathcal{G} \models^+ B$ or $\mathcal{G} \models^+ C$, then $\mathcal{G} \models^+ B \vee C$;*

(h) *if $\mathcal{G} \models^- B$ and $\mathcal{G} \models^- C$, then $\mathcal{G} \models^- B \vee C$;*

(i) *if $\mathcal{G} \models^- B$ or $\mathcal{G} \models^+ C$, then $\mathcal{G} \models^+ B \to C$;*

(j) *if $\mathcal{G} \models^+ B$ and $\mathcal{G} \models^- C$, then $\mathcal{G} \models^- B \to C$;*

(k) *if* $\mathcal{G} \models^+ B$ *and* $\mathcal{G} \models^+ C$, *or* $\mathcal{G} \models^- B$ *and* $\mathcal{G} \models^- C$, *then* $\mathcal{G} \models^+ B \leftrightarrow C$;

(l) *if* $\mathcal{G} \models^+ B$ *and* $\mathcal{G} \models^- C$, *or* $\mathcal{G} \models^- B$ *and* $\mathcal{G} \models^+ C$, *then* $\mathcal{G} \models^- B \leftrightarrow C$;

(m) *if* $\mathcal{G} \models^+ B[x := P]$, *for every term* P, *then* $\mathcal{G} \models^+ \forall x\, B$;

(n) *if* $\mathcal{G} \models^- B[x := P]$, *for some term* P, *then* $\mathcal{G} \models^- \forall x\, B$;

(o) *if* $\mathcal{G} \models^- B[x := P]$, *for every term* P, *then* $\mathcal{G} \models^- \exists x\, B$;

(p) *if* $\mathcal{G} \models^+ B[x := P]$, *for some term* P,*then* $\mathcal{G} \models^+ \exists x\, B$;

(q) *if* $\mathcal{G} \models^+ B[x := P]$, *then* $\mathcal{G} \models^+ P \in \{x \mid B\}$;

(r) *if* $\mathcal{G} \models^- B[x := P]$, *then* $\mathcal{G} \models^- P \in \{x \mid B\}$.

Proof.

(b). If F_i is not of the form $\exists x\, A$ and $\mathcal{G} \not\models^+ F_i$, then $\Gamma_i \vdash F_i, \Delta_i$ is not cut-free derivable and F_i belongs to Δ_{i+1}. Therefore $\Gamma_{i+1}, F_i \vdash \Delta_{i+1}$ is an initial sequent and $\mathcal{G} \models^- F_i$. If F_i is $\exists x\, A$ and $\mathcal{G} \not\models^- F_i$, then $\Gamma_i, F_i \vdash \Delta_i$ is not cut-free derivable and F_i is in Γ_{i+1}. Therefore $\Gamma_{i+1} \vdash F_i, \Delta_{i+1}$ is an initial sequent and $\mathcal{G} \models^+ F_i$.

(i). If $\mathcal{G} \models^- B$ or $\mathcal{G} \models^+ C$, then for some i, $\Gamma_i, B \vdash \Delta_i$ or $\Gamma_i \vdash C, \Delta_i$ is cut-free derivable. Using weakening and rule $\rightarrow_R$, one has that, for some i, $\Gamma_i \vdash B \rightarrow C, \Delta_i$ is cut-free derivable, hence $\mathcal{G} \models^+ B \rightarrow C$.

(j). If $\mathcal{G} \models^+ B$ and $\mathcal{G} \models^- C$ then, for some i, $\Gamma_i \vdash B, \Delta_i$ and for some j, $\Gamma_j, C \vdash \Delta_j$ are cut-free derivable. By rule $\rightarrow_L$ (and weakening), $\Gamma_k, B \rightarrow C \vdash \Delta_k$ is cut-free derivable, where k is the maximum of i and j, hence $\mathcal{G} \models^- B \rightarrow C$.

(m). If $\mathcal{G} \not\models^+ \forall x\, B$, then $\Gamma_i \vdash \forall x\, B, \Delta_i$ is not cut-free derivable, for i such that $\forall x\, B$ is F_i. Since there is a variable y such that $B[x := y]$ belongs to Δ_{i+1}, it is not true, for every term P, that $\mathcal{G} \models^+ B[x := P]$.

(n). If $\mathcal{G} \models^- B[x := P]$, for some term P, then there exists an i such that $\Gamma_i, B[x := P] \vdash \Delta_i$ is cut-free derivable. Hence, by the $\forall_L$-rule, $\Gamma_i, \forall x B \vdash \Delta_i$ is cut-free derivable, i.e. $\mathcal{G} \models^- \forall x\, B$.

(q). If $\mathcal{G} \models^+ B[x := P]$, then for some i, $\Gamma_i \vdash B[x := P], \Delta_i$ is cut-free derivable. Applying the $\in_R$-rule it follows that $\Gamma_i \vdash P \in \{x \mid B\}, \Delta_i$ is also cut-free derivable, i.e. $\mathcal{G} \models^+ P \in \{x \mid B\}$.

The other cases are analogous. □

3 Proof of the "Hauptsatz"

The proof of GENTZEN's theorem will be carried out in NFU with ROSSER's axiom. We describe this system now. The language of NFU (QUINE's NF, possibly with *urelemente*, see [8]) is the language of SF extended by adding the equality symbol. In a weak stratification, the types assigned to P and Q, at occurrences of $P = Q$, ought be the same. $\emptyset$ is the term $\{x \mid \neg x = x\}$, which is intended to denote the "empty set".

In NFU, a set is either a non empty object or the empty set: thus, if present, *urelemente* are not sets. Accordingly, we let $\mathrm{Set}(x)$ abbreviate the following formula: $x = \emptyset \vee \exists y\, y \in x$. We view terms like $\{x \mid A\}$ as denoting sets. The non logical axioms are the obvious axioms for equality and the following ones:

- $\text{Set}(\{x \mid A\})$;
- $\forall x(x \in \{x \mid A\} \leftrightarrow A)$ (comprehension for weakly stratifiable formulas);
- $\forall x \forall y(\text{Set}(x) \wedge \text{Set}(y) \rightarrow (\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y))$ (extensionality for sets).

All the machinery (and more) of type theory is available in this system. As in type theory, cardinal numbers are Frege-Russell's equivalence classes under equinumerousity and natural numbers are cardinals of finite sets. The set of natural numbers is denoted by a closed term Nat.

The well-known type raising operation T is defined as:

$$T(|z|) = |\{\{y\} \mid y \in z\}| \quad \text{and} \quad T(x) = \emptyset, \text{ if } x \text{ is not a cardinal number.}$$

$T(x) = y$ is stratifiable by stratifications that give to y a type one higher than that given to x. ROSSER's axiom[2], which is unstratifiable, says that

$$(\forall y \in \text{Nat})\, |\{x \in \text{Nat} \mid x < y\}| = y$$

or alternatively that $(\forall x \in \text{Nat})\, x = T(x)$. It is well known that this axiom implies the axiom of infinity: $\emptyset \notin \text{Nat}$. We refer to [5] or [7] for further details.[3].

As announced, we will work in NFU plus ROSSER's axiom – hereafter NFUR. This is not so unnatural as one might fear because most can be done in higher-order arithmetic, which is part of NFUR. We need only be careful when we assert the existence of a set to ensure that its defining formula is equivalent to a weakly stratifiable one: since most of the sets used can be defined in type theory the problem occurs very rarely.

We code variables, terms, formulas, sequents and derivations of $\mathcal{L}_{\mathsf{SF}}$ – the language of SF – by natural numbers. The proofs in the first part of the paper are then formalized in higher order Peano arithmetic, whence in NFUR. In particular, there are formulas $\mathcal{G} \models^{+} x$ and $\mathcal{G} \models^{-} x$, depending on the non cut-free derivable $\mathcal{L}_{\mathsf{SF}}$-sequent $\Gamma \vdash \Delta$, such that Lemma 2 holds. Henceforth when we speak of $\mathcal{L}_{\mathsf{SF}}$-terms, $\mathcal{L}_{\mathsf{SF}}$-formulas etc., we will refer to the *codes*. What is relevant in the coding is that $T(x) = x$, for each $\mathcal{L}_{\mathsf{SF}}$-expression x. We will use Wiener-Kuratowski's definition of ordered pairs: $\langle x, y\rangle$ is the term

$$\{z \mid z = \{z \mid z = x\} \vee z = \{z \mid z = x \vee z = y\}\}.$$

$\langle x, y\rangle = z$ is stratifiable with x and y having the same type, and z two types higher.

There are terms $\text{sort}(x)$ of NFUR such that $\text{sort}(x) = P$ if P is an $\mathcal{L}_{\mathsf{SF}}$-term, x is not empty, and every y in x is $\langle z, P\rangle$, for some z; and $\text{sort}(x) = \emptyset$, else. Clearly, ROSSER's axiom implies that $\forall x\, T(\text{sort}(x)) = \text{sort}(x)$. We say that x *is of sort* P iff $\text{sort}(x) = P$, and that x is *sorted* iff there is an $\mathcal{L}_{\mathsf{SF}}$-term P such that x is of sort P. $\text{sort}(x) = y$ is stratifiable with x having the type of y increased by 3. Let Sort be the set of sorted sets. Sort is defined by a stratifiable condition:

$$\text{Sort} = \{x \mid \exists y\, (y \in \text{Nat} \wedge y = \text{sort}(x))\}.$$

We write $\alpha, \beta, \ldots$ to denote sorted sets and $\alpha_P, \beta_Q, \ldots$ to denote sets of sort $P, Q, \ldots$.

[2] It is also called "AxCount" (for "Axiom of Counting").

[3] In [3], we proved that NFU is interpretable in SF.

Fixing a non cut-free derivable sequent and a sequence $\mathcal{G}$ satisfying the clauses in Lemma 2, we write $P \in^{+} Q$ for $\mathcal{G} \models^{+} P \in Q$ and $P \in^{-} Q$ for $\mathcal{G} \models^{-} P \in Q$ and we define a relation $\in_{\mathcal{S}}$ on the sorted sets by stipulating:

$$\alpha_P \in_{\mathcal{S}} \beta_Q \quad \text{iff} \quad P \notin^{-} Q, \text{ or } \langle \alpha_P, Q\rangle \in \beta_Q \text{ and } P \in^{+} Q.$$

Lemma 3.

(a) *If* $\alpha_P \in_{\mathcal{S}} \beta_Q$, *then* $P \in^{+} Q$; *if* $\alpha_P \notin_{\mathcal{S}} \beta_Q$, *then* $P \in^{-} Q$.

(b) $\alpha_P \notin_{\mathcal{S}} \beta_Q$ *iff* $P \notin^{+} Q$, *or* $\langle \alpha_P, Q\rangle \notin \beta_Q$ *and* $P \in^{-} Q$.

Proof. These are easy consequences of the fact that $P \in^{+} Q$ or $P \in^{-} Q$, by Lemma 2. □

A *valuation* is a function from a finite set of $\mathcal{L}_{\mathsf{SF}}$-variables to the sorted sets. If v is a valuation, x an $\mathcal{L}_{\mathsf{SF}}$-variable and α a sorted set, then $v[x \mapsto \alpha]$ is the valuation whose domain is the domain of v, extended if necessary by x, and such that $v[x \mapsto \alpha](x) = \alpha$ and $v[x \mapsto \alpha](y) = v(y)$, for any variable y in its domain other than x. A valuation is said to be *defined* for an $\mathcal{L}_{\mathsf{SF}}$-term or an $\mathcal{L}_{\mathsf{SF}}$-formula iff it is defined for the free variables of that $\mathcal{L}_{\mathsf{SF}}$-term or $\mathcal{L}_{\mathsf{SF}}$-formula. If A is an $\mathcal{L}_{\mathsf{SF}}$-formula and if v is a valuation defined for A, then $A[v]$ is the $\mathcal{L}_{\mathsf{SF}}$-formula obtained by simultaneously substituting in A each free $\mathcal{L}_{\mathsf{SF}}$-variable x by $\mathrm{sort}(v(x))$, and similarly we define $P[v]$. For each formula A and $\mathcal{L}_{\mathsf{SF}}$-term P, given a valuation v defined for it, we define by simultaneous induction $\mathcal{S}, v \models A$, the notion of *satisfaction in the model* $\mathcal{S} = \langle \mathrm{Sort}, \in_{\mathcal{S}}\rangle$, and the *interpretation* $\mathcal{S}(P)(v)$ *of* P *in* $\mathcal{S}$, which is a set of sort $P[v]$. Since most of this definition is standard, we limit ourselves to a few cases:

· $\mathcal{S}(x)(v)$ is $v(x)$, if x is an $\mathcal{L}_{\mathsf{SF}}$-variable;

· $\mathcal{S}, v \models P \in Q$ iff $\mathcal{S}(P)(v) \in_{\mathcal{S}} \mathcal{S}(Q)(v)$;

· $\mathcal{S}, v \models A \rightarrow B$ iff $\mathcal{S}, v \not\models A$ or $\mathcal{S}, v \models B$;

· $\mathcal{S}, v \models \forall x\, A$ iff $\mathcal{S}, v[x \mapsto \alpha] \models A$, for all α in Sort;

· $\mathcal{S}(\{x \mid A\})(v)$ is either the set of ordered pairs $\langle \alpha, \{x \mid A[v]\}\rangle$ such that α is a sorted set and $\mathcal{S}, v[x \mapsto \alpha] \models A$, or the sorted set $\{\langle \emptyset, \{x \mid A[v]\}\rangle\}$, if $\mathcal{S}, v[x \mapsto \alpha] \not\models A$ for all α in Sort.

The key point in our proof is that this set exists since it can be defined by a weakly stratifiable formula in the language of NFU, as the $\mathcal{L}_{\mathsf{SF}}$-formula A is weakly stratifiable. To prove this we remark that $x \in^{+} y$ and $x \in^{-} y$ are stratifiable by giving a same type to x and y, and we consider the formula defining $x \in_{\mathcal{S}} y$ explicitly:

$$\mathrm{sort}(x) \notin^{-} \mathrm{sort}(y) \vee (\langle x, \mathrm{sort}(y)\rangle \in y \wedge \mathrm{sort}(x) \in^{+} \mathrm{sort}(y)).$$

Since, we have $\forall x\, T(\mathrm{sort}(x)) = \mathrm{sort}(x)$, this formula is equivalent to the formula

$$T^3(\mathrm{sort}(x)) \notin^{-} \mathrm{sort}(y) \vee (\langle x, \mathrm{sort}(y)\rangle \in y \wedge T^3(\mathrm{sort}(x)) \in^{+} \mathrm{sort}(y))$$

which is stratifiable with y three types higher than x. So we see, by induction, that if A is weakly stratifiable, then $\mathcal{S}, v \models A$ is equivalent to a weakly stratifiable formula. More precisely, given a weak stratification for A, we see that there is a weak stratification for (a formula equivalent to) $\mathcal{S}, v \models A$ that assigns to an occurrence of the term $v(x)$ the type $3n$ if n was the type assigned to the corresponding occurrence of the $\mathcal{L}_{\mathsf{SF}}$-variable x in the weak stratification for A.[4)]

Substitution Lemma.

(a) $\mathcal{S}, v \models A[x := P]$ *iff* $\mathcal{S}, v[x \mapsto \mathcal{S}(P)(v)] \models A$.

(b) $\mathcal{S}(P[x := Q])(v) = \mathcal{S}(P)(v[x \mapsto \mathcal{S}(Q)(v)])$. □

Persistence Lemma.

(a) *If* $\mathcal{S}, v \models A$, *then* $\mathcal{G} \models^+ A[v]$.

(b) *If* $\mathcal{S}, v \not\models A$, *then* $\mathcal{G} \models^- A[v]$.

The proof is by induction on the length of A. If $\mathcal{S}, v \models P \in Q$, that is $\mathcal{S}(P)(v) \in_{\mathcal{S}} \mathcal{S}(Q)(v)$, then $P[v] \in^+ Q[v]$, by Lemma 3; similarly if $\mathcal{S}, v \not\models P \in Q$, then $P[v] \in^- Q[v]$. If $\mathcal{S}, v \models B \to C$, then $\mathcal{S}, v \not\models B$ or $\mathcal{S}, v \models C$. By the induction hypothesis, $\mathcal{G} \models^- B[v]$ or $\mathcal{G} \models^+ C[v]$, hence $\mathcal{G} \models^+ (B \to C)[v]$, by Lemma 2. If $\mathcal{S}, v \models \exists x\, B$, then $\mathcal{S}, v[x \mapsto \alpha_P] \models B$, for some sorted set α_P. By the induction hypothesis $\mathcal{G} \models^+ B[v[x \mapsto \alpha_P]]$, hence $\mathcal{G} \models^+ \exists x\, B[v]$, by Lemma 2. If $\mathcal{S}, v \not\models \exists x\, B$, then $\mathcal{S}, v[x \mapsto \alpha_P] \not\models B$, for every α_P and the induction hypothesis shows that $\mathcal{G} \models^- B[v[x \mapsto \alpha_P]]$, for every α_P. Now we remark that the "function" sort from the sorted sets in the $\mathcal{L}_{\mathsf{SF}}$-terms is onto: for any $\mathcal{L}_{\mathsf{SF}}$-term P, let can(P) be $\{\langle \emptyset, P\rangle\}$; clearly can$(P)$ is a sorted term of sort P. So we have $\mathcal{G} \models^- B[v[x \mapsto \mathrm{can}(P)]]$, for each P, hence $\mathcal{G} \models^- \exists x\, B[v]$, by Lemma 2. The other cases are handled in a similar way. □

Comprehension Lemma.

$$\mathcal{S}, v \models P \in \{x \mid A\} \textit{ iff } \mathcal{S}, v \models A[x := P].$$

Proof. If $\mathcal{S}, v \models P \in \{x \mid A\}$, that is $\mathcal{S}(P)(v) \in_{\mathcal{S}} \mathcal{S}(\{x \mid A\})(v)$, then if $\mathrm{sort}(\mathcal{S}(P)(v)) \notin^- \mathrm{sort}(\mathcal{S}(\{x \mid A\})(v))$ we have $\mathcal{G} \not\models^- P[v] \in \{x \mid A\}[v]$, and by Lemma 2, $\mathcal{G} \not\models^- A[x := P][v]$, hence, by the Persistence Lemma, $\mathcal{S}, v \models A[x := P]$; else $\langle \mathcal{S}(P)(v), \{x \mid A\}[v]\rangle \in \mathcal{S}(\{x \mid A\})(v)$, which implies, by the Substitution Lemma, that $\mathcal{S}, v \models A[x := P]$, because $\mathcal{S}(P)(v) \neq \emptyset$. Using Lemma 3, a dual argument shows that if $\mathcal{S}, v \not\models P \in \{x \mid A\}$, then $\mathcal{S}, v \not\models A[x := P]$. □

Soundness Lemma. *If the $\mathcal{L}_{\mathsf{SF}}$-sequent $\Gamma \vdash \Delta$ is derivable and if $\mathcal{S}, v \models C$ for every $\mathcal{L}_{\mathsf{SF}}$-formula C in Γ, then $\mathcal{S}, v \models D$ for at least one $\mathcal{L}_{\mathsf{SF}}$-formula D of Δ.*

Proof. This is proved by induction on the length of the derivations, using the Comprehension Lemma for the $\in_L$ and $\in_R$ rules, the definition of $\mathcal{S}, v \models A$ (and the Substitution Lemma) for the other introduction rules and finally the obvious fact that $\mathcal{S}, v \models A$ or $\mathcal{S}, v \not\models A$ for the cut rule. □

Theorem. *Every derivable sequent of* SF *is cut-free derivable.*

Proof. Let v be a canonical valuation such that $v(x) = \{\langle \emptyset, x\rangle\}$ for every $\mathcal{L}_{\mathsf{SF}}$-variable x in its domain. If $\Gamma \vdash \Delta$ is not cut-free derivable, then, by Lemma 2, for every formula C and D in Γ and Δ, respectively, $\mathcal{G} \not\models^- C$ and $\mathcal{G} \not\models^+ D$. Hence, by the Persistence Lemma, $\mathcal{S}, v \models C$ and $\mathcal{S}, v \not\models D$. Therefore, $\Gamma \vdash \Delta$ is not derivable

[4)] The factor 3 can be dropped if one works in NFUR with a type-level ordered pair as urged by HOLMES (see [8]).

(Soundness Lemma). Thus, we have shown in NFUR that if $\Gamma \vdash \Delta$ is not cut-free derivable, then it is not derivable at all. Now, there exists an ω-model of NFUR ([8], [1], [2]). If $\Gamma \vdash \Delta$ is not cut-free derivable, there exists no (code of a) cut-free derivation of it in such a model, the natural numbers being the true ones. Hence, there is no derivation of this sequent in the model. Therefore, $\Gamma \vdash \Delta$ is not derivable in the real world. □

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(Received: October 20, 1993)