Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 29, S. 235–237 (1983)

ON THE REDUCTION OF TYPE THEORY

by MARCEL CRABBÉ in Louvain-la-Neuve (Belgium)

§ 1. QUINE's New Foundations (NF) and type theory (TT) have been reduced to some of their fragments by GRISHIN [4]. These fragments are built up from the extensionality axioms and comprehension axioms using at most four successive types. BOFFA has shown that it is impossible, in the case of NF, to reduce the system to axioms using three types [1]. His proof gives also a similar result for TT: there exists no reduction of TT to a uniform set of axioms which contains three successive types at most [2].

These proofs use GÖDEL's second incompleteness theorem. In this paper, these negative facts are derived from a general result about automorphisms of fragments of types structures that do not extend to global structures. It will also be shown that the restriction on uniformity can be dropped and that the axioms which make use of the first four types are essential.

§ 2. TT is here the theory of types corresponding to QUINE's NF and investigated by SPECKER in [6]. TT_n is the fragment of TT reduced to the first n types: $1, \ldots, n$. L and L_n are the languages in which these theories are written. L is the language of TT and L_n the one of TT_n. A structure $\mathfrak{M}$ for L_n is a $2n-1$-uple: $(M_1, E_1, M_2, E_2, \ldots, M_{n-1}, E_{n-1}, M_n)$, where the M_i's $(1 \leqq i \leqq n)$ are pairwise disjoint sets and, for each i $(1 \leqq i < n)$, E_i is a relation between M_i and M_{i+1}. Similarly, a structure for L is a sequence in which, for each n, the $2n-1$ first terms form a structure for L_n. The fragment $\mathfrak{M}[i, j]$ $(1 \leqq i < j \leqq n)$ of the structure $\mathfrak{M}$ is the structure $(M_i, E_i, \ldots, M_j)$ for L_{1+j-i}. $\mathfrak{M}^+$ is the fragment of $\mathfrak{M}$ obtained by dropping M_1 and the relation E_1 (this cannot be done, of course, if $\mathfrak{M}$ is a structure for L_1).

Notions defined for first order structures can usually be extended to typed structures in a natural manner. For example, two structures $\mathfrak{M}$ and $\mathfrak{M}'$ are *isomorphic* iff there is a sequence $(\ldots, f_i, \ldots)$ of bijections between M_i and M_i' such that for all x in M_i and y in M_{i+1}, xE_iy holds iff $f_i(x)\, E_i' f_{i+1}(y)$ holds. As usual, we write $\mathfrak{M} \equiv \mathfrak{M}'$ when $\mathfrak{M}$ and $\mathfrak{M}'$ are elementarily equivalent structures for the same language.

§ 3. From $\mathfrak{M}$ and an automorphism α $(\alpha = (\alpha_1, \ldots, \alpha_{k-1}))$ of the fragment $\mathfrak{M}[2, k]$, one obtains a structure $\mathfrak{M}^\alpha$ by merely replacing the relation E_1 of $\mathfrak{M}$ by a relation E_α, where $xE_\alpha y$ holds, by definition, iff $xE_1\alpha_1(y)$ holds. It is clear that the function (identity on $M_1, \alpha_1, \ldots, \alpha_{k-1}$) from $\mathfrak{M}^\alpha[1, k]$ to $\mathfrak{M}[1, k]$ is an isomorphism. We thus have the following

Lemma. 1. *If $\varphi(\mathbf{x}^1, \ldots, \mathbf{x}^k)$ is a formula of L_k and if, for each i $(1 \leqq i \leqq k)$, $\mathbf{a}_i$ is a sequence of elements of M_i having the same length as $\mathbf{x}^i$, then*

1. $\mathfrak{M}^\alpha \vDash \varphi(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ *iff* $\mathfrak{M} \vDash \varphi(\mathbf{a}_1, \alpha_1(\mathbf{a}_2), \ldots, \alpha_{k-1}(\mathbf{a}_k))$.
2. $\mathfrak{M}^\alpha[1, k] \equiv \mathfrak{M}[1, k]$.
3. $\mathfrak{M}^{\alpha+} = \mathfrak{M}^+$.

Proposition. 1. *Let* $4 \leqq k \leqq n$, $\mathfrak{M}$ *be a model of* TT_n (*or* TT), $\mathfrak{M}^*$ *be a structure for* L_n (*or* L) *such that* $\mathfrak{M}[i, i+k-1]$ *and* $\mathfrak{M}^*[i, i+k-1]$ *are elementarily equivalent for each* i *such that* $i+k-1 \leqq n$ (*or for each* i). *Then* $\mathfrak{M}^*$ *is also a model of* TT_n (*or* TT).

2. *If* $k < 4 \leqq n$, *then for every model* $\mathfrak{M}$ *of* TT_n (*or* TT) *with an infinite* M_1, *there is a structure* $\mathfrak{M}^*$ *for* L_n (*or* L) *such that* $\mathfrak{M}[1, k] \equiv \mathfrak{M}^*[1, k]$, $\mathfrak{M}^+ \equiv \mathfrak{M}^{*+}$, *but* $\mathfrak{M}^*$ *is not a model of* TT_n (*or* TT).

Proof. The first part is a consequence of Grishin's type reductions. For the second part, we only consider the case where $n = 4$ and $k = 3$. The other cases should be evident. So, we shall prove that, given an infinite model $\mathfrak{M}$ of TT_4, there is a structure $\mathfrak{M}^*$ such that $\mathfrak{M}[1, 3] \equiv \mathfrak{M}^*[1, 3]$ and $\mathfrak{M}[2, 4] \equiv \mathfrak{M}^*[2, 4]$ but $\mathfrak{M}^* \nvDash TT_4$.

First, let's recall some known facts about models of TT_2 (see [3] and [5]). A model $\mathfrak{M}$ of TT_2 is called *countably saturated* if M_1 and M_2 are countably infinite and, for each a in M_2 such that $\{x \in M_1 \mid xE_1a\}$ is infinite, there is a b in M_2 such that the sets $\{x \in M_1 \mid xE_1a \text{ and } xE_1b\}$ and $\{x \in M_1 \mid xE_1a \text{ and not-}xE_1b\}$ are both infinite. Countably saturated models of TT_2 are *homogeneous* (i.e., if $\mathfrak{M}$ and $\mathfrak{M}'$ are two countably saturated models of TT_2 and $\boldsymbol{a}$, $\boldsymbol{b}$ two finite sequences of same length of elements of M_2 and M_2' respectively such that the corresponding bits of $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same cardinality, then there exists an isomorphism from $\mathfrak{M}$ to $\mathfrak{M}'$ mapping each term of $\boldsymbol{a}$ onto the corresponding term of $\boldsymbol{b}$) and *universal* (i.e. every countable model of TT_n (or TT) has an elementary extension $\mathfrak{M}$ such that for every $i < n$ (or for every i), $\mathfrak{M}[i, i+1]$ is countably saturated).

Let $\mathfrak{M}$ be an infinite model of TT_4. We may suppose that $\mathfrak{M}[2, 3]$ is countably saturated. One chooses an element e in M_1 and defines a and b as the elements of M_3 that fulfil the following requirements:

$$\mathfrak{M} \models \forall y^2(y^2 \in a \Leftrightarrow e \in y^2) \quad \text{and} \quad \mathfrak{M} \models b = USC(V_1)$$

($USC(V_1)$ in the usual terminology of type theory is the "set" of all the singletons of the individuals: $\{x^2 \mid \exists x^1\, \forall y^1(y^1 \in x^2 \Leftrightarrow x^1 = y^1)\}$). Since the structure $\mathfrak{M}[2, 3]$ is homogeneous, it has an automorphism α that exchanges a and b. Let $PU(x^3)$ be the formula $\exists z^1\, \forall y^2(y^2 \in x^3 \Leftrightarrow z^1 \in y^2)$. $\mathfrak{M} \models PU(a)$ and, because $PU(x^3)$ is in L_3,

$$(*) \qquad \mathfrak{M}^\alpha \models PU(b)$$

follows from the lemma. Let $IC(x^3)$ be the formula asserting that there is a set of (unordered) pairs establishing a bijection between x^3 and its complement ($\{y^2 \mid y^2 \notin x^3\}$). Through a natural modification of the proof of Cantor's theorem in TT, one gets $\neg IC(USC(V_1))$ as a theorem of TT_4 plus the axiom

$$\exists x^1 \exists y^1 \exists z^1 (x^1 \neq y^1 \wedge x^1 \neq z^1 \wedge y^1 \neq z^1).^{1)}$$

---

[1]) If there is a function f from $USC(V)$ onto its complement, we call C_f the set $\{x \mid x \notin f\{x\}\}$. We claim that C_f is not a singleton when there are at least three individuals. Indeed, if $C_f = \{a\}$ we choose two individuals, b and c, distinct from a. Then, if $f\{x\}$ is $\{a, b\}$ the individual x cannot be a, since $a \notin f\{a\}$. So it must be b, because $x \in f\{x\}$ when $x \neq a$. Thus $f\{b\} = \{a, b\}$. For the same reasons $f\{c\} = \{a, c\}$. But again, if $f\{x\} = \{a, b, c\}$, we have that $x = b$ or c, which contradicts the fact that f is a function. Since C_f belongs to the complement of $USC(V)$, Cantor's proof goes through: there is a d such that $f\{d\} = C_f$; but then $d \in C_f$ iff $d \notin C_f$.

Since the formula $IC(x^3)$ does not mention the type 1, we conclude:

(**) $\mathfrak{M}^\alpha \nvDash IC(b)$.

Now, $\mathfrak{M}[1, 3] \equiv \mathfrak{M}^\alpha[1, 3]$ and $\mathfrak{M}[2, 4] \equiv \mathfrak{M}^\alpha[2, 4]$ as it can be seen from lemma 1. On the other hand, if $\mathfrak{M}^\alpha$ was a model of TT_4, we should have $\mathfrak{M}^\alpha \vDash PU(b) \to IC(b)$, because $\forall x^3(PU(x^3) \to IC(x^3))$ is provable in TT_4. But this contradicts (*) and (**). □

The restriction in the proposition to models with an infinite M_1 is essential because the theory of any model of TT_n (or TT), with a finite M_1 is categorical.

§ 4. TT^∞ is the theory resulting from TT by the addition, for each n, of the sentence asserting that there are at least n individuals ($\exists x_1^1 \ldots \exists x_n^1 \bigwedge_{1 \leq i < j \leq n} x_i^1 \neq x_j^1$). The definition of TT_n^∞ is similar. TT_n^∞ (or TT^∞) is thus the theory of the models $\mathfrak{M}$ of TT_n (or TT) having an infinite M_1. A *3-typed theory* is a theory written in L whose non logical axioms that mention the type 1 are all in L_3. We are now in a position to draw some of the consequences of the proposition above.

Theorem.

1. *No* 3-*typed theory having a model with infinite* M_1 *is an extension of* TT_4.

2. *If* $n > 3$, TT_n^∞ *is not included in a* 3-*typed consistent theory.* TT^∞ *is not included in a* 3-*typed consistent theory.*

3. *If* $n > 3$, TT_n *is not equal to a* 3-*typed theory.* TT *is not equal to a* 3-*typed theory.*

4. NF *is not included in a consistent theory, written in the language of* NF, *all of whose non logical axioms could be stratified with the indices* 1, 2 *and* 3 (Boffa [1]).

References

[1] Boffa, M., On the axiomatization of NF. In: Colloque international de Logique, Clermont-Ferrand 1975, pp. 157–159.

[2] Boffa, M., A reduction of the theory of types. In: Set Theory and Hierarchy Theory V, Lecture Notes in Mathematics **619** (1977), pp. 95–100.

[3] Grishin, V. N., Consistency of a fragment of Quine's NF system. Soviet Math. Dokl. **10** (1969), 1387–1390.

[4] Grishin, V. N., The method of stratification in set theory. Academy of Science of the USSR, Moscow 1973.

[5] Oswald, U., Inequivalence of the fragments of New Foundations. Arch. für Math. Log. **21** (1981), 77–82.

[6] Specker, E., Typical ambiguity. In: Logic, Methodology and Philosophy of Science, Stanford 1962, pp. 116–124.

Marcel Crabbé
Université de Louvain (UCL)
Institut Supérieur de Philosophie
Chemin d'Aristote, 1
1348 Louvain-la-Neuve
– Belgique –

(Eingegangen am 10. November 1981)