On the Set of Atoms

MARCEL CRABBÉ, *Université Catholique de Louvain, Département de Philosophie, Place Mercier, 14, B–1348 Louvain-la-Neuve, Belgium. E-mail: crabbe@risp.ucl.ac.be*

Abstract

We extend to *NFU* the familiar permutation method used in *NF* to obtain independence results for unstratified sentences. We apply this technique in order to characterize the theories resulting from the addition to *NFU* of the supposition that the number of atoms is less or equal to the number of sets or the supposition that the number of atoms is greater or equal to the number of sets. Although we show further that no stratified sentence on sets can be shown independent using this method alone, we prove nevertheless independence results for stratified sentences referring to the set of atoms.

Keywords: NF, NFU, Type Theory, New Foundations, Quine.

1 Introduction

NFU is Quine's New Foundations (usually abbreviated *NF*) with the axiom of extensionality weakened to allow atoms or urelements (see [10], [7], [9], [4] and/or the *NF* home page `math.idbsu.edu/~holmes/holmes/nf.html`).

The language of *NFU* is the usual language of set theory, to which we add an extra constant $\emptyset$ representing the empty set. The non logical axioms of *NFU* are the stratifiable comprehension scheme and the extensionality axiom for non-empty objects. We add $\forall x\, x \notin \emptyset$ to these in order to make explicit the meaning of the symbol $\emptyset$.

From an informal point of view, we may interpret the universe of *NFU* as made up of objects, which are either sets or atoms, the atoms being considered as objects without elements and all the sets save *the* empty set being not empty.

The permutation method[1] for proving the consistency of unstratifiable sentences with *NF* was first introduced by Scott [12] and developed by Henson [8] (see also [7] and, for a general setting, [11]). The idea of using surjective or injective functions in place of bijections in the case of *NFU* or other systems with weak extensionality axioms is already present in Boffa [1] and Crabbé [4].

We will in the next section present a general method of renaming sets which includes the case of permutations in *NF* and other functions in *NFU* and prove the equivalent of Scott's theorem. We will also define a class of stratified formulas and prove the analogue of a result by Henson for these formulas.

In the last section, we give alternative characterizations of the theories *NFU*+ "the number of atoms is less or equal to the number of sets" and *NFU*+ "the number of atoms is greater or equal to the number of sets". We conclude the paper by showing that the sentence expressing that the number of atoms is infinite is not equivalent in *NFU* to a stratified sentence on sets if and only if *NF* itself is consistent.

[1] It should be clear that this technique is *not* the permutation method of Fraenkel and Mostowski.

The following notations will be used as abbreviations to denote sets[2]: V for the universe: $\{ x \mid x = x \}$; U for the set of urelements: $\{ x \mid \forall y\, y \notin x \wedge x \neq \emptyset \}$; $\mathcal{P}(a)$ for the power set of a: $\{ x \mid x \text{ is a subset of } a \}$, where "$x$ is a subset of a" stands for the formula expressing that every element of x is an element of a and that x is a set: $\forall y\,(y \in x \rightarrow y \in a) \wedge (\exists v\, v \in x \vee x = \emptyset)$.

We note that the set of all sets is $\mathcal{P}(V) = \{ x \mid \exists v\, v \in x \vee x = \emptyset \}$; the set of urelements is $U = \{ x \mid x \notin \mathcal{P}(V) \}$ so that $\mathcal{P}(V) = V \setminus U$.

2 Renaming the sets in *NFU*

Suppose there is given a set S and a *bijective* "internal" function[3] s from S to the set of sets: $\mathcal{P}(V) = V \setminus U$. Extend s to a total function s_* of V to $\mathcal{P}(V)$ by letting $s_*(x) = \emptyset$, for $x \in V \setminus S$.

Define:

$$x \in_s y \quad \text{as} \quad x \in s_*(y)$$

and select an empty set in the new "model" by letting $\emptyset^s = s^{-1}(\emptyset)$. Such a s will be called a renaming function or simply a *renaming*[4].

2.1 Renaming functions interpret NFU

We show that the universe with the new relation and the new empty set is an interpretation — a "model" — of *NFU*. In the sequel ϕ^s denotes the result of replacing $\in$ by $\in_s$ and $\emptyset$ by $\emptyset^s$ in (the unabbreviated form of) ϕ.[5]

Let's begin with *stratified comprehension*. Remark first that ϕ^s is stratified when ϕ is. Hence, if ϕ is stratified, there is a y such that:

$$\forall x\,(x \in y \leftrightarrow \phi^s)$$

If such a y is not empty then it is in $\mathcal{P}(V)$, if it is empty we can also have it in $\mathcal{P}(V)$ by taking it to be $\emptyset$. Thus our y being in the range of s, there is a z such that $s_*(z) = y$. So we obtain:

$$\exists z\, \forall x\,(x \in_s z \leftrightarrow \phi^s)$$

The axioms of comprehension are thus verified.

The axiom of *extensionality* of *NFU*, $(\exists w\, w \in x \wedge \forall v\,(v \in x \leftrightarrow v \in y)) \rightarrow x = y$, is interpreted as follows:

$$(\exists w\, w \in_s x \wedge \forall v\,(v \in_s x \leftrightarrow v \in_s y)) \rightarrow x = y$$

But if $\exists w\, w \in_s x$ and $\forall v\,(v \in_s x \leftrightarrow v \in_s y)$, then $x, y \in S$ and thus, by the axiom of extensionality, $s(x) = s_*(x) = s_*(y) = s(y)$. We conclude that $x = y$, because s is one-one.

So, by induction on the length of proofs in *NFU*, we obtain the:

---

[2] Which means that an abstract $\{ x \mid \phi \}$ denotes $\emptyset$ if no x fulfills ϕ.

[3] "Internal" means here that the collection of ordered pairs forming the extension of the function is a set.

[4] So that the sets previously labeled with elements of $\mathcal{P}(V)$ are relabeled with elements of S, things not in the new set of labels being considered as urelements in the new "universe" or "model". Thus $x \in_s y$ is equivalent to $x \in s(y) \wedge y \in S$, giving another way to define $\in_s$.

[5] Thus $[x \in \emptyset]^s$ is $x \in_s \emptyset^s$, which is equivalent to $x \in s_*(\emptyset^s) = \emptyset$; and $[x = \emptyset]^s$ is $x = \emptyset^s$.

Proposition 2.1 *If NFU $\vdash \phi$, then NFU + "s is a renaming function" $\vdash \phi^s$.*

Remarks 2.2 *This proposition was first proved in the case of permutations in NF by Scott [12], who credits Bernays and Rieger for a similar result in other systems.*

If $V = S$ then s is an bijective function from V onto $\mathcal{P}(V)$ and $s_* = s$ is an injective function from V into V.

If we are in NF then, as $\mathcal{P}(V) = V$, s_* is a surjection from V onto V.

Finally, if $S = V = \mathcal{P}(V)$, then $\in_s$ is nothing else than a permutation interpretation in the ordinary sense.

2.2 Invariant Statements

It is well known that stratified sentences are invariant under permutation interpretations. This is no longer true for renaming. Indeed we will see that a sentence about the number of atoms is normally stratifiable and a renaming function typically will stretch or shrink the set of atoms without altering the size of the set of sets. Nevertheless, as we will show in this section, the result remains true if instead of stratified sentences, we consider stratified sentences on sets: i.e. sentences that are about sets, sets of sets,...

The usual notational conventions will be used: if T is or abbreviates $\{ x \mid \phi \}$, T^s stands for $\{ x \mid \phi^s \}$, so that $[x \in T]^s$ is $x \in T^s$.

Lemma 2.3 *In NFU+ "s is a renaming with domain S", one has: $V^s = V$, $\mathcal{P}(V)^s = S$ and $U^s = V \setminus S$.*

Proof

$x \in V^s$ means $(x = x)^s$, i.e. $x = x$.

$x \in \mathcal{P}(V)^s$ means $(\exists y\, y \in x \vee x = \emptyset)^s$, which is $\exists y\, y \in s_*(x) \vee x = \emptyset^s$, that is $x \in S$.

$x \in U^s$ means $x \in V^s \wedge x \notin \mathcal{P}(V)^s$, which is $x \in V \wedge x \notin S$.

⌟

Let as usual $f[x]$ denote the range of the function f restricted to the domain x: $f[x] = \{ f(z) \mid z \in x \}$. We now define a hierarchy of total functions.

Definition 2.4

$$s_0(x) = x$$
$$s_{n+1}(x) = s_n[s_*(x)]$$

Lemma 2.5 *In NFU + "s is a renaming", the sequence of functions $\langle s_0, s_1, s_2, \ldots \rangle$ restricted to the typed structure $\langle V^s, \in_s, \mathcal{P}(V)^s, \in_s, \mathcal{P}(\mathcal{P}(V))^s, \ldots \rangle$ is an isomorphism of:*
$\langle V^s, \in_s, \mathcal{P}(V)^s, \in_s, \mathcal{P}(\mathcal{P}(V))^s, \ldots \rangle$ and $\langle V, \in, \mathcal{P}(V), \in, \mathcal{P}(\mathcal{P}(V)), \ldots \rangle$, such that each s_{n+1} sends $\emptyset^s$ to $\emptyset$.

Proof

Let us stay in *NFU* + "s is a renaming". It is clear that $s_{n+1}(\emptyset^s) = s_n[s_*(\emptyset^s)] = s_n[\emptyset] = \emptyset$ and it is equally clear that s_0 is a bijective function between $\mathcal{P}^0(V)^s = V$ and $\mathcal{P}^0(V) = V$.

Assuming, as inductive hypothesis, that s_n is a bijective function from $\mathcal{P}^n(V)^s$ to $\mathcal{P}^n(V)$, we now show that:

a if $x \in \mathcal{P}^n(V)^s$ and $y \in \mathcal{P}^{n+1}(V)^s$, then:

$$x \in_s y \leftrightarrow s_n(x) \in s_{n+1}(y)$$

b s_{n+1} is a bijective function from $\mathcal{P}^{n+1}(V)^s$ to $\mathcal{P}^{n+1}(V)$.

This will then complete the proof.

Before we proceed, we observe that it is a consequence of proposition 2.1 that:

$$\forall x \forall y \, (y \in \mathcal{P}^{n+1}(V)^s \wedge x \in_s y \to x \in \mathcal{P}^n(V)^s) \tag{2.1}$$

Proof of **a**. Suppose $x \in \mathcal{P}^n(V)^s$ and $y \in \mathcal{P}^{n+1}(V)^s$.

If $x \in_s y$, it is immediate that $s_n(x) \in s_n[s_*(y)] = s_{n+1}(y)$.

For the converse, suppose $s_n(x) \in s_{n+1}(y) = s_n[s_*(y)]$. Then there is a z such that $s_n(z) = s_n(x)$ and $z \in_s y$. Since $y \in \mathcal{P}^{n+1}(V)^s$, we have $z \in \mathcal{P}^n(V)^s$, by (2.1). Therefore, since by the inductive hypothesis s_n is injective on $\mathcal{P}^n(V)^s$, $z = x$ and thus $x \in_s y$ as required.

Proof of **b**. Let x and y be distinct elements of $\mathcal{P}^{n+1}(V)^s$. By proposition 2.1 and (2.1), there is a $z \in \mathcal{P}^n(V)^s$ such that $z \in_s x$ and $z \notin_s y$ (or $z \notin_s x$ and $z \in_s y$). Therefore, by **a**, $s_n(z) \in s_{n+1}(x)$ and $s_n(z) \notin s_{n+1}(y)$ (or $s_n(z) \notin s_{n+1}(x)$ and $s_n(z) \in s_{n+1}(y)$). This proves that the restriction of s_{n+1} to $\mathcal{P}^{n+1}(V)^s$ is injective.

Next we show that $\mathcal{P}^{n+1}(V)$ includes the image under s_{n+1} of $\mathcal{P}^{n+1}(V)^s$. We remark that if $x \in \mathcal{P}^{n+1}(V)^s$, then by (2.1) every $y \in_s x$ is in $\mathcal{P}^n(V)^s$, and, by the inductive hypothesis, $s_n(y) \in \mathcal{P}^n(V)$. Therefore $s_{n+1}(x) = s_n[s_*(x)] \in \mathcal{P}^{n+1}(V)$.

To conclude, we must also show that s_{n+1}, restricted to $\mathcal{P}^{n+1}(V)^s$, is surjective onto $\mathcal{P}^{n+1}(V)$. Let $x \in \mathcal{P}^{n+1}(V)$. By the inductive hypothesis, each $w \in x$ is the image under s_n of an (in fact unique) element of $\mathcal{P}^n(V)^s$, so that $x = s_n[v]$, for some $v \in \mathcal{P}(\mathcal{P}^n(V)^s)$. This v belonging to $\mathcal{P}(V)$ is the image under s of some $X \in S$. Thus $s_{n+1}(X) = s_n[s_*(X)] = s_n[v] = x$. Finally, if $z \in_s X$ then $z \in v$ and this implies that $z \in \mathcal{P}^n(V)^s$. Therefore $X \in \mathcal{P}^{n+1}(V)^s$, because $\forall z \, (z \in_s X \to z \in \mathcal{P}^n(V)^s) \to X \in \mathcal{P}^{n+1}(V)^s$, by proposition 2.1. ⌟

The typed structure $< V, \in, \mathcal{P}(V), \in, \mathcal{P}(\mathcal{P}(V)), \in, \dots >$ can be viewed as a model of type theory. If ϕ is a stratified formula, the formula saying that ϕ is true in this structure is stratified too. This leads to the formal definition:

Definition 2.6 *If ϕ is a stratified formula with a fixed stratification*[6]*, then $V \models \phi$ will denote the stratified formula obtained from ϕ by restricting its quantifiers with index i to $\mathcal{P}^i(V)$. A stratification we use to do this is called* appropriate.

A formula of the form $V \models \phi$ will be called on sets.

Remarks 2.7 *In [2], Boffa introduced the notion of* typed property*: he considers formulas of the form $x \models \phi$, expressing that the stratified sentence ϕ, viewed as a sentence of type theory (a typed sentence), is true in the structure $\langle x, \in, \mathcal{P}(x), \in, \mathcal{P}(\mathcal{P}(x)), \dots \rangle$, formed by the sets above x; and he defines a typed property of x as a formula equivalent in NF (or NFU) to one of the form $x \models \phi$. Thus a sentence on sets expresses the fact that the universe verifies a typed property; this notion of typed property of the universe is nonetheless relative to the system, while that of sentence on sets is not.*

[6] We exclude the negative integers from the range of stratifications.

In NF, every stratified sentence is a typed property of the universe: i.e. is equivalent to a sentence on sets because there $V = \mathcal{P}(V)$ *and* $\forall x\, \phi$ *is equivalent to* $(\forall x \in V)\, \phi$.

Sentences like the axiom of infinity or the axiom of choice are equivalent in NFU to sentences on sets; however the axiom of extensionality, saying that everything is a set, is (perhaps) not equivalent in NFU to a sentence on sets.

Proposition 2.8 *In NFU+ "s is a renaming" one proves that* $x_1 \in \mathcal{P}^{i_1}(V)^s, x_2 \in \mathcal{P}^{i_2}(V)^s, \ldots$, *imply*

$$\phi(x_1, x_2 \ldots)^s \leftrightarrow \phi(s_{i_1}(x_1), s_{i_2}(x_2) \ldots)$$

if ϕ *is a formula on sets and for some appropriate stratification all the* x_1, x_2,... *free in* ϕ *have been assigned respectively the indices* i_1, i_2,...

Proof

This proposition is a restatement of lemma 2.5 and generalizes to renaming functions a result by Henson [8]. ┘

Lemma 2.9 *Let* ψ *be a formula on sets such that for some appropriate stratification the index 1 has been assigned to all the variables* $x_1, \ldots, x_n$ *free in* ψ *and such that, for* $1 \leq i \leq n$, x_i *does not occur free in* χ_i, *then:*

$$[\exists x_1 \ldots x_n \in \mathcal{P}(V)\, (\forall z(z \in x_1 \leftrightarrow \chi_1) \wedge \ldots \wedge \forall z(z \in x_n \leftrightarrow \chi_n) \wedge \psi(x_1 \ldots x_n))]^s$$
$$\leftrightarrow$$
$$\exists x_1 \ldots x_n \in \mathcal{P}(V)\, (\forall z(z \in x_1 \leftrightarrow \chi_1^s) \wedge \ldots \wedge \forall z(z \in x_n \leftrightarrow \chi_n^s) \wedge \psi(x_1 \ldots x_n))$$

is provable in NFU + "s is a renaming".

Proof

Without loss of generality, we deal with the case $n = 1$ only. Using lemma 2.3, proposition 2.8 and the fact that $s_1 = s_*$, $[\exists x \in \mathcal{P}(V)\, (\forall z(z \in x \leftrightarrow \chi) \wedge \psi(x))]^s$ is shown successively equivalent to:

$$\exists x \in \mathcal{P}(V)^s\, (\forall z(z \in s_*(x) \leftrightarrow \chi^s) \wedge \psi(x)^s)$$
$$\exists x \in S\, (\forall z(z \in s_*(x) \leftrightarrow \chi^s) \wedge \psi(s_*(x)))$$
$$\exists x \in \mathcal{P}(V)\, (\forall z(z \in x \leftrightarrow \chi^s) \wedge \psi(x))$$

┘

Remarking that, for any set abstract T, we have $T \in \mathcal{P}(V)$ and $T^s \in_s S$, by proposition 2.1 and lemma 2.3, we obtain the following consequence of lemma 2.9:

Proposition 2.10 *If* $\psi(x_1 \ldots x_n)$ *is on sets with an appropriate stratification attributing the index 1 to* $x_1 \ldots x_n$, *the only variables free in it, and* $T_1, \ldots, T_n$ *are set abstracts, then:*

$$[\exists x_1 \ldots x_n\, (x_1 = T_1 \wedge \ldots \wedge x_n = T_n \wedge \psi(x_1 \ldots x_n))]^s$$
$$\leftrightarrow$$
$$\exists x_1 \ldots x_n\, (x_1 = T_1^s \wedge \ldots \wedge x_n = T_n^s \wedge \psi(x_1 \ldots x_n))$$

is provable in NFU + "s is a renaming"; in short:

$$\text{NFU + "}s \text{ is a renaming"} \vdash \psi(T_1 \ldots T_n)^s \leftrightarrow \psi(T_1^s \ldots T_n^s)$$

┘

Remark 2.11 *If $\psi(x)$ is (equivalent) to a formula on sets, $\psi(T)$ need not be generally so.*

$\exists s\, \psi$ will be used as an abbreviation of $\exists s$ (“s is a renaming” $\wedge\, \psi$).

Theorem 2.12 *If $\psi(x_1 \ldots x_n)$ is a formula on sets with an appropriate stratification assigning 1 to all its free variables $x_1, \ldots, x_n$, and if $T_1, \ldots, T_n$ are set abstracts, then:*

$$NFU + \exists s\, \psi(T_1^s \ldots T_n^s) \vdash \phi \quad \textit{iff} \quad NFU + \psi(T_1 \ldots T_n) \vdash \phi$$

for all sentences on sets ϕ.

Proof

Taking s as the identity function on $\mathcal{P}(V)$, the “only if” direction becomes obvious. So we limit ourselves to the “if” direction.

Let ϕ be a sentence on sets such that $NFU + \psi(T_1 \ldots T_n) \vdash \phi$. By proposition 2.1, we have:

$$NFU + \text{“}s \text{ is a renaming”} \vdash \psi(T_1 \ldots T_n)^s \to \phi^s$$

Combined with proposition 2.10, this entails that:

$$NFU + \text{“}s \text{ is a renaming”} \vdash \psi(T_1^s \ldots T_n^s) \to \phi$$

Therefore:

$$NFU + (\text{“}s \text{ is a renaming”} \wedge \psi(T_1^s \ldots T_n^s)) \vdash \phi$$

and thus $NFU + \exists s\, \psi(T_1^s \ldots T_n^s) \vdash \phi$. ⌟

3 The Number of Atoms

3.1 The axiom of infinity and the number of atoms

Definition 3.1 *AI is the axiom of infinity. This axiom is a 3-stratifiable sentence and is equivalent to a sentence on sets.*

To avoid misunderstanding, we define $\aleph_0$ as the least cardinal of an infinite well orderable set, $\aleph_1$ as the next cardinal of a well orderable set... If no such cardinals exist then $\aleph_0 = \emptyset$, or $\aleph_1 = \emptyset$... Thus $|x| = \aleph_n$ implies AI. We must also modify the natural definition of the sum of cardinals by setting $\kappa + \mu = \emptyset$ when there are no disjoint sets of cardinal κ and μ.

The next theorem characterizes the theorems on sets of *NFU* plus the hypothesis that the number of atoms is less or equal to the number of sets.

Theorem 3.2 *The theory $NFU + |U| \leq |\mathcal{P}(V)|$ has the same (closed) theorems on sets as each of the following: $NF = NFU + |U| = 0$, $NFU + |U| = n$, $NFU + |U| = \aleph_n$ (n a natural number) and $NFU + AI +$ “$|U|$ is finite”.*

Proof

First we show that the following sentences are proved equivalent in $NFU + AI$:

a $\exists x\, (|x| \leq |\mathcal{P}(V)| \wedge |\mathcal{P}(V)| + |x| = |V|)$,

b $|\mathcal{P}(V)| + n = |V|$ (for each natural number n),

c $|\mathcal{P}(V)| + \aleph_n = |V|$ (for each natural number n),

d $\exists x$ ("x is finite" $\wedge\ |\mathcal{P}(V)| + |x| = |V|$).

In *NFU* + *AI*, one proves $\aleph_n \leq |\mathcal{P}^n(V)| \leq |\mathcal{P}(V)|$ ($\aleph_n$ exists, is not empty), and thus $\alpha \leq |\mathcal{P}(V)|$ if α is a finite cardinal or is $\aleph_n$. Therefore, each of **b**, **c**, **d** implies **a**

In order to show that **a** implies **b**, **c** and **d**, we observe first that (in *NFU* + *AI*) we have[7] $|USC(V)| + 1 = |USC(V)|$ and, from this, $|\mathcal{P}(V)| + |\mathcal{P}(V)| = 2^{|USC(V)|} + 2^{|USC(V)|} = 2^{|USC(V)|+1} = 2^{|USC(V)|} = |\mathcal{P}(V)|$. It follows that $|\mathcal{P}(V)| + |x| = |\mathcal{P}(V)|$ iff $|x| \leq |\mathcal{P}(V)|$. Assume **a**, then $|\mathcal{P}(V)| = |V|$ and thus $|\mathcal{P}(V)| + |x| = |V|$ for *every* $|x| \leq |\mathcal{P}(V)|$. Whence **b**, **c** and **d**.

Combining what we just proved with the fact that $|\mathcal{P}(V)|+|U| = |V|$, the following theories are shown identical:

a* *NFU* + *AI* + $\exists s\, |U^s| \leq |\mathcal{P}(V)^s|$,

b* *NFU* + *AI* + $\exists s\, |U^s| = n$,

c* *NFU* + *AI* + $\exists s\, |U^s| = \aleph_n$,

d* *NFU* + *AI* + $\exists s$ "$|U^s|$ is finite".

Remarking that the formulas $|x| \leq |y|$, $|x| = n$, $|x| = \aleph_n$ (for $n \geq 0$) and "$|x|$ is finite" are (equivalent to formulas) on sets, we may conclude, by theorem 2.12, that the following theories have the same theorems on sets:

a** *NFU* + *AI* + $|U| \leq |\mathcal{P}(V)|$,

b** *NFU* + *AI* + $|U| = n$,

c** *NFU* + *AI* + $|U| = \aleph_n$,

d** *NFU* + *AI* + "$|U|$ is finite".

In [6], we prove that, in *NFU*, $|U| \leq |\mathcal{P}(V)|$ implies *AI*. Therefore **a****, **b**** and **c**** are identical to *NFU* + $|U| \leq |\mathcal{P}(V)|$, *NFU* + $|U| = n$ and to *NFU* + $|U| = \aleph_n$ respectively (we remind that $|U| = \aleph_n$ implies *AI*). ┘

Remark 3.3 *The equiconsistency of NF, NFU* + $|U| = 1$, *NFU* + $|U| = 2$,... *and the equiconsistency of NF and NFU* + *AI* + $|U| \leq |\mathcal{P}(V)|$ *were already proved in [1] and [3]. M. Boffa drew our attention to the fact that the concluding remark of [2] easily entails that NFU* + $|\mathcal{P}(V)| = |V|$ *has the same theorems on sets as NF.*

Theorem 3.4 *The theories NFU* + $|U| \geq |\mathcal{P}(V)|$ *and NFU have the same (closed) theorems on sets.*

Proof

By theorem 2.12, *NFU* + $|U| \geq |\mathcal{P}(V)|$ has the same theorems on sets as *NFU* + $\exists s\, |U^s| \geq |\mathcal{P}(V)^s|$. It will suffice thence to prove in *NFU* that $\exists s\, |U^s| \geq |\mathcal{P}(V)^s|$.

NFU+*AI* proves that there is a set S such that $|S| = |\mathcal{P}(V)|$ and $|V \setminus S| = |\mathcal{P}(V)| + |U|$. This is because $|\mathcal{P}(V)| + |\mathcal{P}(V)| + |U| = |V|$ is provable in it. $\exists s\, |U^s| \geq |\mathcal{P}(V)^s|$ is thus provable in *NFU* + *AI*.

---

[7] $USC(V)$ is the set of all the singletons.

On the other hand *NFU* $+ \neg AI$ proves $|U| \geq |\mathcal{P}(V)|$ because it proves $|U| \leq |\mathcal{P}(V)| \vee |U| \geq |\mathcal{P}(V)|$ and, as shown in [6], $|U| \not\leq |\mathcal{P}(V)|$. $\exists s\, |U^s| \geq |\mathcal{P}(V)^s|$ is thus provable in *NFU* $+ \neg AI$ too.

Hence *NFU* proves $\exists s\, |U^s| \geq |\mathcal{P}(V)^s|$. ┘

3.2 How to say that the set of atoms is infinite?

Definition 3.5 *NFU∞ is NFU plus the sentences stating that there are at least* 1, 2, ..., n, ... *urelements.*

UI is the sentence expressing that there is an infinity of urelements. It is 4-stratifiable (but maybe not equivalent to a sentence on sets).

Theorem 3.6 *Let T be any extension of NFU that is a subtheory of NFU$\infty + AI$. Then:*
NF is consistent iff
UI is not equivalent in T to a sentence on sets iff
UI is not equivalent in T to a 3-stratifiable sentence.

Proof

We first assume that *NF* is inconsistent. By theorem 3.2, *NFU* $+ |U| \leq |\mathcal{P}(V)|$ is inconsistent too. But clearly, *NFU* $+ AI + \neg UI \vdash |U| \leq |\mathcal{P}(V)|$. Therefore *NFU* $+ AI + \neg UI$ is inconsistent and *NFU* $\vdash AI \rightarrow UI$. On the other hand, *NFU* $\vdash UI \rightarrow AI$. Therefore in *NFU*, and thus in T, *UI* is equivalent to a 3-stratifiable sentence, namely *AI*, and also to a sentence on sets.

Now we prove the converse. Let's suppose that there is a sentence on sets ψ such that $NFU\infty + AI \vdash \psi \leftrightarrow UI$. Then $NFU\infty + AI + UI \vdash \psi$. Since $NFU + AI + |U| \geq |\mathcal{P}(V)| \vdash NFU\infty + AI + UI$, we have $NFU + AI + |U| \geq |\mathcal{P}(V)| \vdash \psi$ and, observing that *AI* is equivalent to a sentence on sets, theorem 3.4 yields $NFU + AI \vdash \psi$. Thus $NFU\infty + AI \vdash UI$. It follows that, for some natural number n, $NFU + |U| \leq n + AI \vdash UI$. Hence $NFU + AI \vdash |U| > n$. But since, from the celebrated result by Specker [13], *NFU*$+$*AI* is a subtheory of *NF*, *NF* proves that there are atoms and so is inconsistent.

In order to prove this in the 3-stratifiable case, we use the following proposition (Crabbé [5]): *Every 3-stratifiable sentence is equivalent in NFU∞ to a sentence on sets.*

If *UI* was equivalent in $NFU\infty + AI$ to a 3-stratifiable sentence, then it would also be equivalent to a sentence on sets, and, by what we just proved, *NF* would be inconsistent .

┘

Remarks 3.7 *Note that, by [13], UI is equivalent in NFU $+$ AC to AI.*

The result can be generalized in the following way: if Γ is a set of sentences on sets (or 3-stratifiable sentences), then for any theory T between NFU $+ \Gamma$ and NFU$\infty + \Gamma + AI$, UI is equivalent in T to a sentence on sets (or to a 3-stratifiable sentence) iff NF $+ \Gamma$ is inconsistent.

References

[1] Boffa, M. *Entre NF et NFU*. Comptes rendus de l'académie des sciences de Paris, Volume 277, série A, p. 821–822, 1973.

[2] Boffa M., *Sets Equipollent to their Power Set in NF*. The Journal of Symbolic Logic, Volume 40, number 2, p. 149–150, 1975.

[3] Boffa, M. *The Consistency Problem for NF*. The Journal of Symbolic Logic, Volume 42, number 2, p. 215–220, 1977.

[4] Crabbé, M. *On NFU*. Notre Dame Journal of Formal Logic, Volume 33, number 1, p. 112–119, 1992.

[5] Crabbé, M. *The 3-Stratifiable Theorems of NFU∞*. to appear in Notre Dame Journal of Formal Logic.

[6] Crabbé, M. *L'axiome de l'infini dans NFU*. Comptes rendus de l'Académie des Sciences de Paris, Volume 329, série I, p. 1033–1035, 1999.

[7] Forster, Th. *Set Theory with a Universal Set. Exploring an Untyped Universe.* Second edition. Oxford Logic Guides n° 31, Clarendon Press, Oxford, 1995.

[8] Henson, C. W. *Permutations Methods applied to NF*. The Journal of Symbolic Logic, Volume 38, number 1, p. 69–76, 1973.

[9] Holmes, M. R. *Elementary Set Theory with a Universal Set.* Cahiers du Centre de Logique, Volume 10, Academia-Bruylant, Louvain-la-Neuve, 1998.

[10] Jensen, R. B. *On the Consistency of a Slight* (?) *Modification of NF*. Synthese, Volume 19, p. 250–263, 1968–69.

[11] Pétry, A. *Stratified Languages.* The Journal of Symbolic Logic, Volume 57, number 4, p. 1366–1376, 1992.

[12] Scott, D. *Quine's Individuals.* Logic, Methodology and Philosophy of Science, edited by E. Nagel, P. Suppes and A. Tarski, Stanford University Press, Stanford, 1962, pp. 111–115.

[13] Specker E., *The Axiom of Choice in Quine's "New Foundations for Mathematical Logic"*. Proceedings of the National Academy of Science of the United States of America, Volume 39, p. 972–975, 1953.

Received 16 February 2000. Revised 16 May 2000