Notre Dame Journal of Formal Logic
Volume 40, Number 2, Spring 1999

The 3-Stratifiable Theorems of $NFU\infty$

MARCEL CRABBÉ

Abstract It is shown that the 3-stratifiable sentences are equivalent in *NFU* to truth-functional combinations of sentences about objects, sets of objects, sets of sets of objects, and sentences stating that there are at least n urelements. This is then used to characterize the closed 3-stratifiable theorems of *NFU* with an externally infinite number of urelements, as those that can be nearly proved in *TTU* with an externally infinite number of urelements. As a byproduct we obtain a rather simple demonstration of the consistency of 3-stratifiable extensions of *NFU*.

1 Introduction *NFU* is Quine's *NF* with the axiom of extensionality weakened to allow urelements (see Jensen [8], Forster [5], Holmes [7], and Crabbé [3]). We will suppose for convenience that the language of *NFU* (language of set theory) includes a constant $\varnothing$ and that besides the specific axioms of *NFU* (stratifiable comprehension and extensionality for nonempty objects) we have the axiom $\forall x\, x \notin \varnothing$. Similarly, *TTU*, the corresponding theory of types will be formulated with constants $\varnothing^1$, $\varnothing^2$, and so on, and axioms $\forall x^0\, x^0 \notin \varnothing^1$, $\forall x^1\, x^1 \notin \varnothing^2$, and so on. This is harmless since these new theories are conservative extensions of the previous ones. We will employ subsequently the standard notation E^+ to denote the result of raising the type superscripts by 1 in the expression E.

2 The 3-stratifiable sentences of NFU It will be necessary in this section to deal with the 3-stratifiable sentences of *NFU* not directly but via the associated type theory *TTU* or better TTU_3, that is, the fragment of *TTU* reduced to the first three types.

The following simple observation can serve as a guideline for understanding the definitions and proofs below. We can distinguish in a model of TTU_3 three parts that are somehow glued together. First a model of TT_3 (TTU_3 with full extensionality) constituted by the objects of type 0, the sets of objects of type 0, and the sets of sets of objects of type 0. Then a structure (a model, if there is an urelement in type 1) for TT_2, constituted by the urelements in type 1 and the sets of those urelements. Finally,

Received September 2, 1999; revised February 4, 2000

a structure (a model, if there is an urelement in type 2) for TT_1, constituted by the urelements in type 2. The objects in types 0 and 1 of the original structure all come from the substructures. The objects of type 2 are either already in the substructures or are unions of objects of type 2 from the first and of type 1 from the second substructure.

Conversely, this paper will make it apparent that, given models of TT_3, TT_2 (or an empty structure), and TT_1 (or an empty structure), we can amalgamate them together in the same way and obtain a model of TTU_3.

We will investigate this phenomenon at the level of sentences thus showing that a sentence about the universe of TTU_3 is equivalent to a truth-functional combination of sentences each of which is about one of the three parts indicated.[1]

2.1 Restricted formulas Let us use the following abbreviations.

1. $\mathsf{Set}^1(x^1)$ for $\exists v^0\, v^0 \in x^1 \vee x^1 = \varnothing^1$, that is, x^1 is a set in type 1;
2. $\mathsf{Set}^2(x^2)$ for $\exists v^1\, v^1 \in x^2 \vee x^2 = \varnothing^2$, which is $\mathsf{Set}^1(x^1)^+$, that is, x^2 is a set in type 2;
3. $\mathsf{HSet}^2(x^2)$ for $\mathsf{Set}^2(x^2) \wedge \forall v^1\, (v^1 \in x^2 \to \mathsf{Set}^1(v^1))$, that is, x^2 is a hereditary set in type 2;
4. $\mathsf{U}^1(x^1)$ for $\neg\mathsf{Set}^1(x^1)$, that is, x^1 is an urelement in type 1;
5. $\mathsf{U}^2(x^2)$ for $\neg\mathsf{Set}^2(x^2)$, that is, x^2 is an urelement in type 2;
6. $\mathsf{USet}^2(x^2)$ for $\mathsf{Set}^2(x^2) \wedge \forall v^1\, (v^1 \in x^2 \to \mathsf{U}^1(v^1))$, that is, x^2 is a set of urelements in type 2.

Definition 2.1 A *restricted quantification* of a formula φ is a formula of the form $\exists x^i\, (\psi(x^i) \wedge \varphi)$ or $\forall x^i\, (\psi(x^i) \to \varphi)$, where $\psi(x^i)$ is either $\mathsf{Set}^1(x^1)$, $\mathsf{HSet}^2(x^2)$, $\mathsf{U}^1(x^1)$, $\mathsf{U}^2(x^2)$, or $\mathsf{USet}^2(x^2)$. The class of *restricted* formulas is the smallest class containing the atomic formulas and closed under truth-functional operations and restricted quantification.

Lemma 2.2 *In TTU_3, every formula is equivalent to a restricted formula without new free variables or new types.*

Proof: (a) We suppose that the notions of "restricted quantification over type 2" and of "formula with quantification over type 2 restricted" are defined in the obvious way and show first, by induction, that every formula is TTU_3-equivalent to a formula in which all quantification over type 2 is restricted.

Suppose that $\exists x^2\, \varphi$ is a formula such that in φ quantification over type 2 is restricted. Since $\mathsf{Set}^2(x^2) \vee \mathsf{U}^2(x^2)$ is an instance of the excluded middle, $\exists x^2\, \varphi$ is equivalent to

$$\exists x^2\, (\mathsf{U}^2(x^2) \wedge \varphi) \vee \exists x^2\, (\mathsf{Set}^2(x^2) \wedge \varphi).$$

We are thus left with the task of proving that $\exists x^2\, (\mathsf{Set}^2(x^2) \wedge \varphi)$, which is not restricted, is equivalent to a formula with restricted quantification over type 2.

This part of the proof will be based on the fact that an object of type 2 that is not an urelement is the union of a set of sets of objects of type 0 and of a set of urelements.

Let $x^2 = v^2 \cup w^2$ abbreviate $\forall z^1\, (z^1 \in x^2 \longleftrightarrow z^1 \in v^2 \vee z^1 \in w^2)$. Then,

$$\forall x^2\, (\mathsf{Set}^2(x^2) \longleftrightarrow \exists v^2 \exists w^2\, (\mathsf{HSet}^2(v^2) \wedge \mathsf{USet}^2(w^2) \wedge x^2 = v^2 \cup w^2)$$

is provable in TTU_3.

Now, consider the formula

$$\exists x^2\,(\exists v^2\,(\mathsf{HSet}^2(v^2) \wedge \exists w^2\,(\mathsf{USet}^2(w^2) \wedge x^2 = v^2 \cup w^2 \wedge \varphi))).$$

This formula—which contains still an unrestricted quantification over type 2—is thus proved equivalent to $\exists x^2\,(\mathsf{Set}^2(x^2) \wedge \varphi)$ by using the axioms of TTU_3.

Finally, it is also equivalent to

$$\exists v^2\,(\mathsf{HSet}^2(v^2) \wedge \exists w^2\,(\mathsf{USet}^2(w^2) \wedge \varphi^*))$$

where φ^* is obtained from φ, by replacing atomic subformulas of the kind $y^1 \in x^2$ by $y^1 \in v^2 \vee y^1 \in w^2$, and atomic formulas of the kind $x^2 = y^2$ or $y^2 = x^2$ by $y^2 = v^2 \cup w^2$. So our initial formula $\exists x^2\,\varphi$ is equivalent to a formula in which all quantifications over type 2 are restricted. The case of the universal quantifier is handled in a analogous way and the remaining cases are trivial.

(b) We conclude in showing, by induction again, that a formula with quantifications over type 2 restricted is TTU_3-equivalent to a restricted formula. Suppose that $\forall x^1\,\varphi$ is a formula such that φ is a restricted formula. Using the fact that $\mathsf{Set}^1(x^1) \vee \mathsf{U}^1(x^1)$ is provable, we see that

$$\forall x^1\,(\mathsf{Set}^1(x^1) \to \varphi) \wedge \forall x^1\,(\mathsf{U}^1(x^1) \to \varphi)$$

satisfies our requirements. The other cases are similar or trivial. □

2.2 Meaningful formulas The metasymbols $\mathsf{Set}^1, \mathsf{HSet}^2, \mathsf{U}^1, \mathsf{U}^2, \mathsf{USet}^2$ will be called *sorts*. A sort assignment to the variables $\vec{x}$ is a conjunction of formulas of the kind $\mathsf{Set}^1(x^1)$, $\mathsf{HSet}^2(x^2)$, $\mathsf{U}^1(x^1)$, $\mathsf{U}^2(x^2)$, or $\mathsf{USet}^2(x^2)$, where each $\vec{x}$-variable occurs in exactly one such formula.

If **S** is a sort assignment to the free variables of a restricted formula φ, we say that, relatively to **S**, a free variable x has sort S if and only if $S(x)$ is one of the conjuncts of **S**; that a bound variable in φ has sort S if and only if its binding quantifier is restricted to $S(x)$.

Formally, the sorts are assigned to variables only. On an informal level, however, we may think of $\varnothing^1$ as having sort Set^1 and of $\varnothing^2$ as having *both* sort HSet^2*and* USet^2.

Definition 2.3 An atomic formula A is called *meaningful* relatively to a sort assignment if and only if

1. A is $x^0 \in y^1$ and y^1 has sort Set^1; or
2. A is $x^1 \in y^2$ and x^1 and y^2 have either sorts Set^1 and HSet^2, or U^1 and USet^2; or
3. A is $\varnothing^1 \in x^2$ and x^2 has sort HSet^2; or
4. A is $x^i = y^i$ and x^i and y^i have same sort; or
5. A is $x^1 = \varnothing^1$ or $\varnothing^1 = x^1$ and x^1 has sort Set^1; or
6. A is $x^2 = \varnothing^2$ or $\varnothing^2 = x^2$ and x^2 has sort HSet^2 or USet^2.

A formula of TTU_3 is *meaningful* relatively to a sort assignment if and only if it is restricted and all its atomic subformulas are meaningful relatively to the sort assignment. A sentence (of TTU_3) is *meaningful* if and only if it is meaningful relatively to the empty conjunction.

Lemma 2.4 *Let* **S** *be a sort assignment to the free variables of a restricted formula* φ*, then there is a meaningful formula* ψ *relatively to* **S** *such that*

$$TTU_3 \vdash \mathbf{S} \rightarrow (\varphi \longleftrightarrow \psi).$$

Moreover, the free variables and types in ψ *already occur in* φ*.*

Proof: The inductive proof consists in replacing the meaningless atomic subformulas in φ with equivalent meaningful ones. If φ is $x^1 \in y^2$ and x^1 has sort U^1 and y^2 has sort HSet^2, we replace it by $\neg x^1 = x^1$. We have

$$TTU_3 \vdash \mathsf{U}^1(x^1) \wedge \mathsf{HSet}^2(y^2) \rightarrow (x^1 \in y^2 \longleftrightarrow \neg x^1 = x^1)$$

by definition of $\mathsf{U}^1(x^1)$ and $\mathsf{HSet}^2(y^2)$, and the result follows.

If φ is $x^1 = y^1$ and x^1 has sort U^1 and y^1 has sort Set^1, we replace it by $\neg x^1 = x^1$. We have

$$TTU_3 \vdash \mathsf{U}^1(x^1) \wedge \mathsf{Set}^1(y^1) \rightarrow (x^1 = y^1 \longleftrightarrow \neg x^1 = x^1).$$

The atomic sentences $\varnothing^1 \in \varnothing^2$, $\varnothing^1 = \varnothing^1$, and $\varnothing^2 = \varnothing^2$ are replaced by $\exists x^1\, (\mathsf{Set}^1(x^1) \wedge \neg x^1 = x^1)$, $\forall x^1\, (\mathsf{Set}^1(x^1) \rightarrow x^1 = x^1)$, and $\forall x^2\, (\mathsf{HSet}^2(x^2) \rightarrow x^2 = x^2)$, respectively.

If φ is $x^2 = y^2$ and x^2 has sort HSet^2 and y^2 has sort USet^2, we replace it by $x^2 = \varnothing^2 \wedge y^2 = \varnothing^2$. This again works because

$$TTU_3 \vdash \mathsf{HSet}^2(x^2) \wedge \mathsf{USet}^2(y^2) \rightarrow (x^2 = y^2 \longleftrightarrow x^2 = \varnothing^2 \wedge y^2 = \varnothing^2).$$

In the remaining cases we replace similarly the meaningless atomic formulas by false meaningful ones with no new variable or new type.

The induction step is an exercise in predicate calculus. For example, if φ is $\exists x^i\, (S(x^i) \wedge \chi)$, with $i \neq 0$, then, by the inductive hypothesis, we have

$$TTU_3 \vdash \mathbf{S} \wedge S(x^i) \rightarrow (\chi \longleftrightarrow \mu).$$

Therefore,

$$TTU_3 \vdash \mathbf{S} \rightarrow (\exists x^i\, (S(x^i) \wedge \chi) \longleftrightarrow \exists x^i\, (S(x^i) \wedge \mu)).$$

□

Combining Lemma 2.2 and 2.4, we obtain the following corollary.

Corollary 2.5 *Every sentence is equivalent in* TTU_3 *to a meaningful sentence.*

2.3 Connected formulas Let us say that two variables in a formula, in which no two quantifiers bind a same variable, are *immediately connected* if and only if they occur in the same atomic subformula and define the relation of connection among variables in a formula as the transitive closure of this relation of immediate connection. A formula is said to be connected if and only if any two variables (free or bound in it) are connected. The technique of renaming bound variables allows us to extend the definition to arbitrary formulas.

The following, less appealing but more handy, equivalent definition will also be used. I take it from Crabbé [2].

Definition 2.6 $\varphi \wedge \psi$ $[\varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \varphi \longleftrightarrow \psi]$ is a connected conjunction [disjunction, implication, equivalence] if and only if there is at least a common free variable in both φ and ψ. $\exists x\, \varphi$ $[\forall x\, \varphi]$ is a connected quantification if and only if x occurs free in φ. The class of connected formulas is the smallest class containing the atomic formulas and closed under negation, connected conjunction, connected disjunction, . . . and connected quantification.

One of the fundamental properties of connected formulas is contained in this proposition.

Proposition 2.7 *Every formula is equivalent to a truth-functional combination of connected formulas without new free variables or new types.*

Proof: I adapt the inductive proof of [2]. The only nontrivial cases are $\exists x^i\, \varphi$ and $\forall x^i\, \varphi$ where φ is assumed to be a truth-functional combination of the connected formulas $\varphi_1, \ldots, \varphi_n$. We limit ourselves to the existential case. Write φ as a disjunction of conjunctions of the $\varphi_1, \ldots, \varphi_n$ or their negations. It is then clear that, distributing $\exists x^i$ within the disjunction, $\exists x^i\, \varphi$ is equivalent to a disjunction of conjunctions prefixed with $\exists x^i$. We arrange these conjunctions so that the conjuncts with occurrences of x^i are grouped together. We may then move $\exists x^i$ inside each conjunction as far as possible or remove it when quantification is vacuous. We thus rewrite the prefixed conjunction as a conjunction of connected formulas in which x^i does not occur free and of a connected quantification that has the form of a prefixed conjunction of formulas in which x^i occurs free. □

Lemma 2.8 *Every meaningful formula is equivalent to a truth-functional combination of connected meaningful formulas without new free variables or new types.*

Proof: We can almost reproduce the proof of the proposition with restricted quantification in place of usual quantification without losing meaningfulness. The only difference being that of "vacuous quantification" in case x^i is not free in χ, we are not allowed to replace $\exists x^i\, (S(x^i) \wedge \chi)$ by $\exists x^i\, S(x^i) \wedge \chi$ because it is not restricted. However, we may use $\exists x^i\, (S(x^i) \wedge x^i = x^i) \wedge \chi$ instead. □

Corollary 2.5 and Lemma 2.8 entail the following.

Corollary 2.9 *Every sentence is equivalent in TTU_3 to a truth-functional combination of connected meaningful sentences.*

Definition 2.10 A *sentence on sets* is a meaningful sentence with type 1 and type 2 quantification restricted to $\mathsf{Set}^1(\ldots)$ and $\mathsf{HSet}^2(\ldots)$. $\mathsf{Ur}_1, \mathsf{Ur}_2, \ldots \mathsf{Ur}_n, \ldots$ denote the (typed with 0 and 1) sentences stating that there are at least $1, 2, \ldots n \ldots$ urelements in type 1. We may write these without the constants $\varnothing^1$, $\varnothing^2$: Ur_n is taken to say that there are at least $n+1$ empty objects of type 1, since we do not consider the emptyset as an urelement.

Lemma 2.11 *A connected meaningful sentence is either a sentence on sets, or a restricted sentence (typed at most with 1, 2) with quantifiers restricted to* $\mathsf{U}^1(\ldots)$ *or to* $\mathsf{USet}^2(\ldots)$*, or a restricted sentence (typed with 2) with quantifiers restricted to* $\mathsf{U}^2(\ldots)$.

Proof: In a meaningful sentence a variable of type 0, or with sort Set^1 or HSet^2, can only be immediately connected with a variable of type 0, a variable of sort Set^1, or a variable of sort HSet^2. This property of the relation of immediate connection extends directly to the general relation of connection. Similarly, a variable with sort U^1 or USet^2 can only be connected with variables of sort U^1 or USet^2; and a variable with sort U^2 is only connected with variables of the same sort. □

Lemma 2.12 *Every connected meaningful sentence is provably equivalent in* TTU_3 *to a sentence on sets or to a truth-functional combination of the* Ur_n *or* Ur_n^+.

Proof: In view of Lemma 2.11, it will be sufficient to show that a connected formula that is not on sets is TTU_3-equivalent to a truth-functional combination of the Ur_n or Ur_n^+. It is easy to show for sentences restricted to $\mathsf{U}^2(\ldots)$. Lazy readers can deduce it from the remainder anyway.

Sentences with quantifiers restricted to $\mathsf{U}^1(\ldots)$ or $\mathsf{USet}^2(\ldots)$ can be seen as sentences about atomic Boolean algebras. The result follows then from the well-known quantifier elimination result of Tarski for Boolean algebras, namely, that a closed theorem of the theory of atomic Boolean algebras is equivalent to a truth-functional combination of sentences saying that there are at least n atoms. □

Definition 2.13 We use E^k as an abbreviation for $E^{++\cdots+}$, where the type raising operation is iterated k times. E^k is termed *E raised by k*. Being careful with respect to bound variables, an expression E of type theory becomes a stratifiable expression denoted $\overline{E}$ by omitting the type superscripts. Thus, $\overline{E}$ and $\overline{E^+}$ are identical up to the names of bound variables. Accordingly, we may define a *sentence on sets* in *NFU* as a sentence $\overline{\varphi}$ for φ a sentence on sets in TTU_3.

We are now in a position to sum up and to import the results obtained for TTU_3 within *TTU* and *NFU*.

Theorem 2.14 *In* TTU_3*, every sentence is equivalent to a truth-functional combination of the* Ur_n, Ur_n^+ *and of sentences on sets. In TTU, every sentence, typed with k, k+1, k+2 at most, is equivalent to a truth-functional combination of the* Ur_n^k, Ur_n^{k+1} *and of sentences on sets raised by k. In NFU, every 3-stratifiable sentence is equivalent to a truth-functional combination of the* $\overline{\mathsf{Ur}_n}$ *and of sentences on sets.*

3 The 3-stratifiable theorems of $NFU\infty$

Definition 3.1 $NFU\infty$ is NFU plus $\overline{\mathsf{Ur}_n}$ for every n. $TTU\infty$ is TTU plus Ur_n^m for every $n, m \geq 0$.

Warning: *$TTU\infty$ is not TTU* plus an externally infinite number of objects of type 0 nor is $NFU\infty$ the theory NFU plus an externally infinite number of objects, which is in fact NFU.

Lemma 3.2 *Every sentence, typed with $k, k+1, k+2$ at most, is provably equivalent in $TTU\infty$ to a sentence on sets raised by k. Every 3-stratifiable sentence is provably equivalent in $NFU\infty$ to a sentence on sets.*

Proof: This is a corollary to Theorem 2.14 since $TTU\infty$ proves each Ur_n^m and $NFU\infty$ proves each $\overline{\mathsf{Ur}_n}$. □

Theorem 3.3 *Let φ be a sentence in the language of TTU_3. Then*

$$NFU\infty \vdash \overline{\varphi} \quad \textit{iff} \quad TTU\infty \vdash \bigvee_{0\leq i<k} \varphi^i \qquad \textit{for some } k.$$

Let Σ, φ be a set of sentences in the language of TTU_3. Then

$$NFU\infty + \overline{\Sigma} \vdash \overline{\varphi} \quad \textit{iff} \quad TTU\infty + \Sigma + \Sigma^+ \cdots \vdash \bigvee_{0\leq i<k} \varphi^i \qquad \textit{for some } k.$$

Proof: As the second part of the theorem derives immediately from the first, we will be concerned only with this one. Forgetting the types, a proof of $\bigvee_{0\leq i<k} \varphi^i$ in $TTU\infty$ translates into a proof of $\bigvee_{0\leq i<k} \overline{\varphi^i}$; that is, of $\overline{\varphi} \vee \cdots \vee \overline{\varphi}$, in NFU. Thus we concentrate on the "only if" part.

Lemma 3.2 enables us to suppose that φ is a sentence on sets. By well-known theorems of Specker and Grishin ([10], [6], see also [5]), if $NFU\infty \vdash \overline{\varphi}$, then

$$TTU_4\infty + \psi_1 \longleftrightarrow \psi_1^+, \ldots, \psi_n \longleftrightarrow \psi_n^+ \vdash \varphi$$

for some sentences $\psi_1, \ldots, \psi_n$, in the language of TTU_3. Let us again assume that these ψ_i are about sets.

Now, suppose that the conclusion of the lemma is false. Then there is a model $\mathcal{M}$ of $TTU\infty + \neg\varphi + \neg\varphi^+ + \neg\varphi^{++} + \cdots + \neg\varphi^{2^n} \ldots$. Since there are exactly 2^n conjunctions of n sentences and whose ith conjunct is either ψ_i or $\neg\psi_i$, we may apply the pigeonhole principle[2] and obtain numbers k and q such that $k < q \leq 2^n$ and that $\mathcal{M}$ satisfies $\psi_1^k \longleftrightarrow \psi_1^q, \ldots, \psi_n^k \longleftrightarrow \psi_n^q$.

Define, as in Boffa [1], a relation $\in_{kq}$ between M_k and M_q as follows:

$$x \in_{kq} y \qquad \text{iff} \qquad \{x\}^{q-k-1} \in_{q-1} y$$
and y is a set of $(q-k-1)$-fold singletons of elements of M_k.

Then the substructures of

$$\langle M_k, \in_k, M_{k+1}, \in_{k+1}, M_{k+2} \rangle \quad \text{and} \quad \langle M_k, \in_{kq}, M_q, \in_q, M_{q+1} \rangle$$

constituted by the objects, the sets, and the sets of sets are isomorphic. Therefore, the structure

$$\langle M_k, \in_{kq}, M_q, \in_q, M_{q+1}, \in_{q+1}, M_{q+2} \rangle$$

extracted from $\mathcal{M}$ is a model of $TTU_4\infty + \neg\varphi + \neg\varphi^+ + \psi_1 \longleftrightarrow \psi_1^+, \ldots, \psi_n \longleftrightarrow \psi_n^+$. Hence, $TTU_4\infty + \psi_1 \longleftrightarrow \psi_1^+, \ldots, \psi_n \longleftrightarrow \psi_n^+ \nvdash \varphi$ and the hypothesis of our *reductio* is false. □

Corollary 3.4 *The theories $NFU\infty + \overline{\Sigma}$ and $TTU\infty + \Sigma + \Sigma^+ \cdots$ are equiconsistent whenever Σ is a set of sentences of the language of TTU_3.*

4 Concluding remarks It is easy to realize that Theorem 3.3 is false with *NFU* in place of $NFU\infty$. The reason for this is that *NF* is $NFU + \neg\overline{\mathsf{Ur}_1}$. Indeed, by the celebrated refutation of the axiom of choice (*AC*) in *NF* (Specker [9]), we have

$$NFU \vdash \neg\overline{\mathsf{Ur}_1} \rightarrow \neg\overline{AC}.$$

But, for no k, $TTU \vdash \bigvee\!\!\!\bigvee_{0 \le i < k} \neg \mathsf{Ur}_1{}^i \rightarrow \neg AC^i$.

We will now indicate why Theorem 3.3 is false when the restriction to 3 types is dropped. In [1], Boffa shows how to obtain a model of *NF* from a model of *NFU* verifying the axiom of infinity (AI) and the sentence "U (the set of urelements) can be mapped injectively into $V \setminus U$".

His proof, combined with the fact that *AC* is refutable in *NF*, shows also that the 4-stratifiable sentence

$$\overline{AI} \wedge \overline{U \text{ is finite}} \rightarrow \neg\overline{AC}$$

is provable in *NFU*, whence in $NFU\infty$. On the other hand, starting with a model of *TT* containing a nonstandard natural number and verifying AI and AC at each level, it is not very hard to build a model of $TTU\infty$ where AI, AC, and "U is finite" are all true at every level.

NOTES

1. The general frame of the proof will be very close to the one used by Dzierzgowski [4] in establishing a totally different result.

2. Readers acquainted with Jensen [8] or Boffa [1] will observe that we do not use Ramsey's theorem at this point, so that we will obtain a very simple proof of the consistency of *NFU* as a corollary.

REFERENCES

[1] Boffa, M., "The consistency problem for *NF*," *The Journal of Symbolic Logic*, vol. 42 (1977), pp. 215–20.

[2] Crabbé, M., "Ambiguity and stratification," *Fundamenta Mathematicæ*, vol. 101 (1978), pp. 11–17.

[3] Crabbé, M., "On *NFU*," *Notre Dame Journal of Formal Logic*, vol. 33 (1992), pp. 112–19.

[4] Dzierzgowski, D., "Équivalence élémentaire de structures stratifiées," *Cahiers du Centre de Logique*, vol. 6 (1986), pp. 47–62.

[5] Forster, T., *Set Theory with a Universal Set, Exploring an Untyped Universe*, 2d edition, Oxford Logic Guides, Clarendon Press, Oxford, 1995.

[6] Grishin, V. N., "The method of stratification in set theory," in Russian, Ph.D. thesis, Moscow University, 1972.

[7] Holmes, M. R., *Elementary Set Theory with a Universal Set*, (Cahiers du Centre de Logique, vol. 10), Academia-Bruylant, Louvain-la-Neuve, 1998.

[8] Jensen, R. B., "On the consistency of a slight (?) modification of *NF*," *Synthese*, vol.19 (1968–69), pp. 250–63.

[9] Specker, E., "The axiom of choice in Quine's 'New foundations for mathematical logic'," *Proceedings of the National Academy of Science of the United States of America*, vol. 39 (1953), pp. 972–75.

[10] Specker, E., "Typical ambiguity," pp. 116–24 in *Logic, Methodology and the Philosophy of Science*, edited by E. Nagel, P. Suppes, and A. Tarski, Stanford University Press, Stanford, 1962.

Université catholique de Louvain
Département de Philosophie
Place Mercier, 14
B-1348 Louvain-la-Neuve
BELGIUM
email: crabbe@risp.ucl.ac.be